SciencePG
Science Publishing Group

On a New Class of Regular Doubly Stochastic Processes

Reza Farhadian[*], Nader Asadian

Department of Mathematics and Statistics, Lorestan University, Khorramabad, Lorestan, Iran

Email address:

farhadian.reza@yahoo.com (R. Farhadian), asadian.n@lu.ac.ir (N. Asadian)
[*]Corresponding author

Abstract: In this article, we show that the well-known Helmert matrix has strong relationship with stochastic matrices in modern probability theory. In fact, we show that we can construct some stochastic matrices by the Helmert matrix. Hence, we introduce a new class of regular and doubly stochastic matrices by use of the Helmert matrix and a special diagonal matrix that is defined in this article. Afterwards, we obtain the stationary distribution for this new class of stochastic matrices.

Keywords: Helmert Matrix, Stochastic Matrix, Markov Chain, Transition Probability, Stationary Distribution, Regular Chain, Ergodic Chain

1. Introduction

In modern probability theory and dynamical systems, stochastic processes and Markov chains are applied contexts that are used in advanced sciences. Two basic topics in stochastic process are *prediction* and *filtering*. In the topic of prediction for Markov chains, we can obtain the k-step transition probability, using the 1-step transition probability. This work is done by a matrix which is called the *stochastic or probability or transition or Markov matrix*. In stochastic processes or Markov chains, stochastic matrices are used for showing the transition probabilities [9, 11]. On the other hand, there is a special matrix in liner algebra that is called the *Helmert matrix*. A Helmert matrix of order n is a square matrix that was introduced by H. O. Lancaster in 1965 [4]. Usually, the Helmert matrix is used in mathematical statistics for analysis of variance (ANOVA), see [1, 2, 8]. In this article, we will show that the Helmert matrix can be used in stochastic processes. For the next sections, the following notation will be used:

(a) I_n denotes an identity matrix of order n.

(b) J_n denotes an $n \times n$ matrix whose elements are all 1.

(c) A_n^{-1} denotes the inverse of a matrix A_n.

(d) A_n^T denotes the transpose of a matrix A_n.

(e) $Diag[a_{11} \quad a_{22} \quad \cdots \quad a_{nn}]$ denotes an $n \times n$ diagonal matrix with diagonal entries $a_{11}, a_{22}, \dots, a_{nn}$.

(f) $A_n > 0$ stands for a matrix A_n all of whose elements are positive.

(g) $\mathbb{R}$ denotes real numbers.

2. Definitions and Particulars

2.1. Stochastic Matrix

Suppose that a stochastic process start from state i to state j. This transition shown by $i \rightarrow j$, and p_{ij} denote its probability. Now, if process consist of n states and X_t denotes the state at time t, then the transition $i \rightarrow j$ at time t, is indicated by $X_t = i$ and $X_{t+1} = j$. Furthermore, a process is called a *Markov chain* if the transition probability p_{ij} is independent of time t for every states i and j of state space. Hence, the transition probability under the *Markov property*, is defined as:

$$p_{ij} = Pr\{X_{t+1} = j | X_t = i, X_{t-1} = i_{t-1}, \dots, X_1 = i_1\}$$

$$= Pr\{X_{t+1} = j | X_t = i\} \qquad (1)$$

for every $i_1, i_2, \dots, i_{t-1}, i, j$ of state space. Thus, we can construct a $n \times n$ stochastic (transition) matrix by the p_{ij}, $1 \leq i, j \leq n$:

Definition 1. *[9] A $n \times n$ real matrix such as $P_n = [p_{ij}]_{n \times n}$ is called a stochastic matrix (or row stochastic matrix), if*

1) $p_{ij} \geq 0, 1 \leq i, j \leq n$

2) $\sum_{j=1}^{n} p_{ij} = 1, \ \forall i \in \{1, 2, \ldots, n\}$

Definition 2. *[5] A doubly stochastic matrix, is a square matrix of nonnegative real numbers with each row and column summing to 1(in other words, a stochastic matrix is doubly, if its transpose is stochastic matrix).*

Hence, If $\left[p_{ij}\right]_{n \times n}$ be the 1-step transition matrix, then the matrix $\left[p_{ij}^{(k)}\right]_{n \times n}$ is called the k-step transition matrix (under the conditions of Definition 1). Chapman and Kolmogorov independently showed that if P is a 1-step transition matrix, then k-step transition matrix denoted by $P^{(k)}$ and is equal to

$$P^{(k)} = P^k = \underbrace{P \times P \times \ldots \times P}_{k \ times} \qquad (2)$$

Or in other words, if ℓ be the state space of a Markov chain, then

$$p_{ij}^{(k)} = Pr\{X_{t+k} = j | X_t = i\} = \sum_{z \in \ell} p_{iz}^{(r)} p_{zj}^{(k-r)} \quad (3)$$

where $0 < r < k$. For more details about the Chapman-Kolmogorov equation, see [9, 10, 11].

2.2. Regular and Ergodic Markov Chains and Stationary Distribution

A nonnegative square matrix P is called *regular* if $P^m > 0$ for some m [6]. Since every transition matrix is nonnegative, hence we have a similar definition for Markov chains.

Definition 3. *[7] A Markov chain is called a regular chain if some powers of the transition matrix has only positive elements (in other words, the transition matrix of chain be regular).*

Regularity is an important property for Markov chains since it has a strong relationship with another important *ergodicity* property. We know that a Markov chain is ergodic if it is possible to go from every state to every state (not necessarily in one move). By [3, Theorem 1.8] we know that if a Markov transition matrix P is regular, then it has exactly one ergodic class and in general this process is ergodic. Hence, the following proposition shows the relationship between the regularity property and ergodicity one:

Proposition 1. *[3, 6] Every regular Markov chain is ergodic.*

Ergodic Markov chains are very important because there is a unique *stationary distribution vector* for their states. To compute the stationary distribution of an ergodic Markov chain, we have:

Proposition 2. *[9] Let $p_{ij}^{(k)}$ be the k-step transition probability from state i to state j and $\lim_{k \to \infty} p_{ij}^{(k)}$ exists. If π_j denotes the stationary distribution and chain be ergodic, then $\pi_j = \lim_{k \to \infty} p_{ij}^{(k)}$.*

Since every regular chain is ergodic, if chain be regular, then stationary distribution exists.

2.3. The Helmert Matrix

The Helmert matrix is a square matrix of order n that is defined as:

$$H_n = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & & \frac{1}{\sqrt{n}} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{i(i-1)}} & \frac{1}{\sqrt{i(i-1)}} & \cdots & \frac{1}{\sqrt{i(i-1)}} & \frac{-(i-1)}{\sqrt{i(i-1)}} & 0 & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{bmatrix}_{n \times n} \qquad (4)$$

Also see [2, page 67].

Moreover, the first row of the Helmert matrix of order n, has the following form

$$\underbrace{\left[\begin{matrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{matrix}\right]}_{n \ items} \qquad (5)$$

And the other i-th rows ($2 \le i \le n$) are formed by

$$\left[\underbrace{\frac{1}{\sqrt{(i-1)i}} \quad \frac{1}{\sqrt{(i-1)i}} \quad \cdots \quad \frac{1}{\sqrt{(i-1)i}}}_{i-1 \ items} \quad \frac{-(i-1)}{\sqrt{(i-1)i}} \quad \underbrace{0 \ \ldots \ 0}_{n-i \ items}\right] \qquad (6)$$

Furthermore, we know that the Helmert matrix is orthogonal [1]:

$$H_n H_n^T = H_n^T H_n = I_n \qquad (7)$$

To prove the main theorems in the next section, we need the following proposition:

Proposition 3. *Let H_n be the Helmert matrix of order n and $D_n = Diag[\gamma + n\beta \quad \gamma \quad \gamma \quad \cdots \quad \gamma]$ with $\gamma, \beta \in \mathbb{R}$. Then*

$$H_n^T D_n H_n \ = \gamma I_n + \beta J_n \qquad (8)$$

Proof. By calculation we have

$$H_n^T \times D_n = \gamma \times H_n^T + \begin{bmatrix} \frac{n\beta}{\sqrt{n}} & 0 & \ldots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n\beta}{\sqrt{n}} & 0 & \ldots & 0 \end{bmatrix}_{n \times n} \qquad (9)$$

Besides, by calculation, we have

$$\beta \times J_n \times H_n^T = \begin{bmatrix} \frac{n\beta}{\sqrt{n}} & 0 & \ldots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n\beta}{\sqrt{n}} & 0 & \ldots & 0 \end{bmatrix}_{n \times n} \qquad (10)$$

Hence, using (10) in (9), we obtain

$$H_n^T \times D_n = \gamma \times H_n^T + \beta \times J_n \times H_n^T \qquad (11)$$

Right multiplying of both sides of (11) by H_n, we obtain

$$H_n^T \times D_n \times H_n = \gamma \times I_n + \beta \times J_n$$

The theorem is proved.

3. Main Results

3.1. A New Class of Regular Doubly Stochastic Matrices Using the Helmert Matrix

Let us consider the matrix

$$P_n = H_n{}^T D_n H_n \qquad (12)$$

where H_n is the Helmert matrix of order n and $D_n = Diag\left[1 \quad \frac{a}{a+nb} \quad \frac{a}{a+nb} \quad \cdots \quad \frac{a}{a+nb}\right]$ with $a, b \geq 0$. Clearly, the matrix P_n is a *diagonalizable matrix* (note that in matrix theory, we know that if a square matrix of order n such as A_n can be equal to $B_n{}^T D_n B_n$ where B_n is a orthogonal matrix and D_n is a diagonal matrix, then A_n is called a diagonalizable matrix (see [12])).

Now, we shall prove that P_n is a doubly stochastic matrix. We need the following lemma.

Lemma 2. *Let H_n be the Helmert matrix of order n and $a, b \geq 0$ such that at least a or b is positive. Then*

$$H_n{}^T D_n H_n = \frac{1}{a+nb}(aI_n + bJ_n) \qquad (13)$$

where $D_n = Diag\left[1 \quad \frac{a}{a+nb} \quad \frac{a}{a+nb} \quad \cdots \quad \frac{a}{a+nb}\right]$.

Proof. Using Proposition 3 with $\gamma = \frac{a}{a+nb}$ and $\beta = \frac{b}{a+nb}$, the lemma is proved.

Theorem 1. *Let $D_n = Diag\left[1 \quad \frac{a}{a+nb} \quad \frac{a}{a+nb} \quad \cdots \quad \frac{a}{a+nb}\right]$ for $a, b \geq 0$ (at least a or b is positive) and H_n be the Helmert matrix of order n. Then $P_n = H_n{}^T D_n H_n$ is a doubly stochastic matrix.*

Proof. By Lemma 2, we know that $P_n = H_n{}^T D_n H_n = \frac{1}{a+nb}(aI_n + bJ_n)$. So, we have

$$P_n = \frac{1}{a+nb}\left(a \overbrace{\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n\times n}}^{I_n} + b \overbrace{\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n\times n}}^{J_n} \right)$$

$$= \begin{bmatrix} \frac{a+b}{a+nb} & \frac{b}{a+nb} & \cdots & \frac{b}{a+nb} \\ \frac{b}{a+nb} & \frac{a+b}{a+nb} & \cdots & \frac{b}{a+nb} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b}{a+nb} & \frac{b}{a+nb} & \cdots & \frac{a+b}{a+nb} \end{bmatrix}_{n\times n} \qquad (14)$$

Clearly, all elements of the above matrix are positive. Now, we shall prove that each row of (14) sum to 1. Hence, we have

For the 1-th row: $\frac{a+b}{a+nb} + \underbrace{\frac{b}{a+nb} + \frac{b}{a+nb} + \cdots + \frac{b}{a+nb}}_{n-1\ items} = \frac{a+nb}{a+nb} = 1$

For the 2-th row: $\frac{b}{a+nb} + \frac{a+b}{a+nb} + \underbrace{\frac{b}{a+nb} + \cdots + \frac{b}{a+nb}}_{n-2\ items} = \frac{a+nb}{a+nb} = 1$

$$\vdots$$

For the i-th row:

$\underbrace{\frac{b}{a+nb} + \frac{b}{a+nb} + \cdots + \frac{b}{a+nb}}_{i-1\ items} + \frac{a+b}{a+nb} +$

$\underbrace{\frac{b}{a+nb} + \frac{b}{a+nb} + \cdots + \frac{b}{a+nb}}_{n-i\ items} = (n-1)\frac{b}{a+nb} + \frac{a+b}{a+nb} = \frac{a+nb}{a+nb} = 1$

$$\vdots$$

For the n-th row: $\underbrace{\frac{b}{a+nb} + \frac{b}{a+nb} + \cdots + \frac{b}{a+nb}}_{n-1\ items} + \frac{a+b}{a+nb} = \frac{a+nb}{a+nb} = 1$

So, both conditions of Definition 1 are holds for the matrix $P_n = H_n{}^T D_n H_n$, hence this matrix is a stochastic matrix.

On the other hand, we know that the matrix P_n is a symmetric matrix, because

$$P_n{}^T = \left(H_n{}^T D_n H_n\right)^T = H_n{}^T D_n H_n = P_n \qquad (15)$$

Thus, by Definition 2, it is a doubly stochastic matrix.

Corollary 1. *Let $\{X_t : t \geq 0\}$ be the finite Markov chain by stochastic matrix P_n in Theorem 1. Then the transition probability for transition $i \rightarrow j$ is equal to*

$$p_{ij} = Pr\{X_{t+1} = j | X_t = i\} = \begin{cases} \frac{a+b}{a+nb} & i = j \\ \frac{b}{a+nb} & i \neq j \end{cases} \qquad (16)$$

Proof. We know that p_{ij} is the (i,j)-th element of stochastic matrix P_n. Therefore the proof is immediately.

Suppose that a square $n \times n$ doubly stochastic matrix such as Q_n has the following form

$$Q_n = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}_{n\times n} = \begin{bmatrix} s & t & \cdots & t \\ t & s & \cdots & t \\ \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & s \end{bmatrix}_{n\times n} \qquad (17)$$

Clearly, since $q_{ii} = s$ and $q_{ij} = t$ (for $i \neq j$), then the matrix Q_n is follow of new class, if $s \geq t$ and the following system of equations hold:

$$\begin{cases} s = \frac{a+b}{a+nb} \\ t = \frac{b}{a+nb} \end{cases} \Rightarrow a, b \geq 0 \text{ and at least } a \text{ or } b \text{ is positive} \qquad (18)$$

Hence we can write $Q_n = \frac{1}{a+nb}(aI_n + bJ_n)$.

Example 1. Consider the matrix $Q_3 = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{2}{7} & \frac{3}{7} \end{bmatrix}_{3\times 3}$. We can show that Q_3 is a doubly stochastic matrix and follow of the new class. Since if it is considered $= \frac{3}{7}$, $t = \frac{2}{7}$ and $n = 3$, then by (18) we have

$$\begin{cases} \frac{3}{7} = \frac{a+b}{a+3b} \\ \frac{2}{7} = \frac{b}{a+3b} \end{cases} \Rightarrow a = 1, b = 2$$

Since both $a = 1$ and $b = 2$ are positive, so, by (18) the matrix Q_3 is follow of the new class.

Now, consider the following another example.

Example 2. Let $Q_2 = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}_{2 \times 2}$. Clearly, Q_2 is a doubly stochastic matrix. But Q_2 is not follow of new class, because if $s = \frac{2}{5}$ and $t = \frac{3}{5}$, then we know that $s < t$, and this is in contradiction with (17) and (18).

Corollary 2. *Let Q_n is a doubly stochastic matrix such that it is hold under the (18). Then Q_n is diagonalizable and $Q_n = H_n^T D_n H_n$ where H_n is the Helmert matrix of order n and $D_n = Diag \left[1 \quad \frac{a}{a+nb} \quad \frac{a}{a+nb} \quad \cdots \quad \frac{a}{a+nb} \right]$.*

Proof. By Corollary 1 if (18) holds for the matrix Q_n, then Q_n is follow of the new class and $Q_n = \frac{1}{a+nb}(aI_n + bJ_n)$. On the other hand, by Lemma 2, we know that the matrix $Q_n = \frac{1}{a+nb}(aI_n + bJ_n)$ is diagonalizable and we can write $Q_n = \frac{1}{a+nb}(aI_n + bJ_n) = H_n^T D_n H_n$ where H_n is the Helmert matrix of order n and D_n is the same diagonal matrix in Lemma 2.

When a matrix is diagonalizable, we can obtain its positive integer powers by its diagonal form. On the other hand, by Chapman-Kolmogorov equation [9, 10, 11], we know that the k-step transition matrix is equal to the k-th power of 1-step transition matrix. Hence, for the stochastic matrix Q_n in Corollary 2, we have

$$Q_n^{\ k} = H_n^T D_n^{\ k} H_n \qquad (19)$$

In the next part, we will use of (19) to compute the stationary distribution of the new class of stochastic matrices.

3.2. Stationary Distribution for the New Class of Regular Stochastic Matrices

We know that the Markov chain by stochastic matrix in the Theorem 1, is a regular and ergodic chain, since all elements in this matrix are positive and hence is possible to go from every state to every state. Now, in the next theorem we compute the stationary distribution for this chain.

Lemma 3. *Let H_n be the Helmert matrix of order n and $D_n = Diag[1 \quad 0 \quad 0 \quad \cdots \quad 0]$. Then $H_n^T D_n H_n = \frac{1}{n} J_n$.*

Proof. The lemma is proved by Lemma 2 with $a = 0$.

Theorem 2. *The Markov chain by the stochastic matrix in Theorem 1 is a regular and ergodic chain, and if π_j (j=1,2,...,n) denotes the stationary distribution of this chain, then $\pi_j = \frac{1}{n}$.*

Proof. By Theorem 1 we know that all elements of stochastic matrix P_n are positive. So, the chain is regular. Besides, by Proposition 1, the chain must be ergodic. By Proposition 2, we know that if chain is ergodic and $\lim_{k \to \infty} p_{ij}^{(k)}$ exists, then $\pi_j = \lim_{k \to \infty} p_{ij}^{(k)}$. So, we have

$$\text{Lim}_{k \to \infty} P_n^{\ k} = \lim_{k \to \infty} \left(H_n^T D_n H_n \right)^k \qquad (20)$$

We know that $D_n = Diag \left[1 \quad \frac{a}{a+nb} \quad \frac{a}{a+nb} \quad \cdots \quad \frac{a}{a+nb} \right]$. Using (19), we have

$$\lim_{k \to \infty} \left(H_n^T \times \begin{bmatrix} 1 & & & \\ & \frac{a}{a+nb} & & O \\ & & \ddots & \\ & O & & \frac{a}{a+nb} \end{bmatrix}_{n \times n} \times H_n \right)^k$$

$$= \lim_{k \to \infty} H_n^T \times \begin{bmatrix} 1 & & & \\ & \left(\frac{a}{a+nb}\right)^k & & O \\ & & \ddots & \\ & O & & \left(\frac{a}{a+nb}\right)^k \end{bmatrix}_{n \times n} \times H_n$$

$$= H_n^T \times \begin{bmatrix} 1 & & & \\ & \lim_{k \to \infty} \left(\frac{a}{a+nb}\right)^k & & O \\ & & \ddots & \\ & O & & \lim_{k \to \infty} \left(\frac{a}{a+nb}\right)^k \end{bmatrix}_{n \times n} \times H_n$$

$$= H_n^T \times \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \times H_n \qquad (21)$$

And by Lemma 3, we have

$$H_n^T \times \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \times H_n = \frac{1}{n} J_n \qquad (22)$$

Therefore $\pi_j = \frac{1}{n}$.

4. Conclusion

Usually, we can working on many topics of probability theory, but working on stochastic processes and offering new Markov chains are less than other subjects. In this article we presented a new class of stochastic matrices. Since for every integer n and every positive real numbers such as a and b, we can construct a stochastic matrix of order n by forming to $P_n = \frac{1}{a+nb}(aI_n + bJ_n)$, so the new class is very big. Also, we showed that $P_n = H_n^T D_n H_n$ where H_n is the well-known Helmert matrix and D_n is a diagonal matrix of order n. So, the stochastic matrix P_n is a diagonalizable matrix. In matrix theory, diagonalizable form of a square matrix is very important, because compute of determinant and also integer powers of matrix by this form is simpler than other usual methods. On the other hand, we know that integer powers of a stochastic matrix is important for prediction of its behavior. In addition, we showed that the Markov chain by the stochastic matrix P_n is regular and ergodic. Regular and ergodic properties are two important properties of stochastic processes, because the stationary distribution for any regular or ergodic Markov chain is equal to the limit of k-th power of its transition matrix as $k \to \infty$. Furthermore, we proved that the stochastic matrix P_n is a doubly stochastic matrix.

Appendix: Doubly Stochastic Matrices that Their Inverses are Doubly Stochastic Matrices

We know that I_n is a doubly stochastic matrix, since each row and column of I_n sum to 1. Besides, $I_n^{-1} = I_n$, which means that there is at least a doubly stochastic matrix of order n such that its inverse is a doubly stochastic matrix. In 2015, R. Farhadian (the first author) by a Farsi language article titled "Approximation for stationary distribution of ergodic stochastic processes" published in "NEDA: Student Statistical Journal", showed that there exists some doubly stochastic matrices except identity matrix, such that their inverses are doubly stochastic matrices. Consider the following matrices of order 2 and 3:

$$M_2 = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{H_2^T} \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{H_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
<center>Diagonalized form</center>

and

$$M_3 = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}}_{H_3^T} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}}_{H_3}$$
<center>Diagonalized form</center>

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where H_2 and H_3 are orthogonal Helmert matrices of order 2 and order 3, respectively.

We know that for a diagonalizable square matrix such as $A_n = B_n^T D_n B_n$, the inverse of A_n is equal to $A_n = B_n^T D_n^{-1} B_n$. Thus for inverses of M_2 and M_3, we have

$$M_2^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} (1)^{-1} & 0 \\ 0 & (-1)^{-1} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$M_3^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix} \times \begin{bmatrix} (1)^{-1} & 0 & 0 \\ 0 & (-1)^{-1} & 0 \\ 0 & 0 & (1)^{-1} \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly, M_2 and M_3 are doubly stochastic matrices and $M_2^{-1} = M_2$ and $M_3^{-1} = M_3$ are doubly stochastic matrices.

References

[1] B. R. Clarke. Linear models: the theory and application of analysis of variance. John Wiley & Sons, New Jersey, 2008.

[2] J. E. Gentie, Numerical linear algebra for application in statistics, Springer, New York, 1998.

[3] A. Kehagias, Approximation of stochastic processes by hidden Markov models, U. M. I, 1992.

[4] H. O. Lancaster. The Helmert matrices. Amer. Math. Monthly, 72 (1965), no. 1, 4-12.

[5] E. Parzen, Stochastic processes, San Francisco: Holden-Day, Inc., 1962.

[6] P. Perkins, A theorem on regular matrices. Pacific J. Math. 11 (1961), no. 4, 1529-1533.

[7] Ch. M. Grinstead, J. L. Snell, Introduction to probability, American Mathematical Society, 1997.

[8] G. A. F. Seber, A matrix handbook for statistician , John Wiley & Sons, New Jersey, 2007.

[9] S. M. Roos, A first cours in probability, 6th ed, Pearson Prentice Hall, 2002.

[10] S. M. Ross, Introduction to probability models. 7th ed. San Diego, colif: Academic Press, Inc., 2000.

[11] S. M. Ross, Stochastic processes. 2nd ed. John wiley & Sons, Inc., New York, 1996.

[12] X. Zhan, Matrix theory, Graduate studies in mathematics, volume 147, American Mathematical Society, 2013.