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Profile Likelihood Confidence Intervals for the Parameters of a Nonhomogeneous Poisson Process with Linear Rate

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Abstract: Nonhomogeneous Poisson Processes (NHPP) are commonly used to model count data where the rate of occurrence of events in a given time period is dependent on time. Examples exist in the literature where NHPP has been used to model real life count data for the purpose of parameter estimation and prediction. The most common methods used to obtain the point estimates of the parameters of the NHPP are the method maximum likelihood and the ordinary least squares method. The commonly used Wald-type confidence intervals are based on the assumption of asymptotic normality and are inaccurate when this assumption is violated This study considers an alternative method based the profile likelihood function to construct approximate confidence intervals for the parameters of a nonhomogeneous Poisson process with linear rate $\lambda(t)=\alpha+\beta t$, based on the number of counts in measurement subintervals. Such a linear rate function is applicable in situations where piecewise-linear approximation to a general rate function is adequate. The profile likelihood confidence intervals for the two parameters are constructed from the graphs of their respective relative profile likelihood functions, which are obtained numerically from the joint relative likelihood function. Simulations were used to compare the profile likelihood and Wald confidence intervals on the basis of coverage probability and mean length. The effects of sample size (number of subintervals) on the interval estimates of the parameters were also investigated. The results of the simulation study show that the profile likelihood method is superior to the Wald method since it yields shorter confide intervals containing plausible values of each of the two parameters.

Keywords: Relative Likelihood Function, Profile Likelihood Function, Profile Likelihood Confidence Intervals, Wald Confidence Intervals

1. Introduction

Nonhomogeneous Poisson process are often applied in modeling counts of events whose rate of occurrence is dependent on time. Nonhomogeneous Poisson process has been used by Sumiati, I. et al. [2] to predict and count the number of earthquakes in Indonesia, [4] to model different kinds of accidents number, and by a number of scientists and engineers to describe software reliability growth models. The rate or intensity function $\lambda(t)$ of Nonhomogeneous Poisson process is a nonnegative integrable function of time. In this project we assume that we have a nonhomogeneous Poisson process over the interval $(0,T)$ with intensity function

$$\lambda(t)=\alpha+\beta t, 0<t<T \tag{1}$$

The method of maximum likelihood is the most common method to estimate the parameters of nonhomogeneous Poisson process of different types of intensity functions and has been considered in [1, 3, 5-8, 14, 15]. Another approach to estimating the parameters of a nonhomogeneous Poisson process is by the method of ordinary least squares (OLS) [1-3].

Confidence intervals are an important tool in statistical inference which quantifies the variability of an estimator of a parameter of interest. Thus, in parametric estimation, a complete estimation statement is obtained when a point estimate is accompanied by a confidence interval. The most common method of constructing confidence interval is called the Wald procedure, which relies on the asymptotic normality of the MLE [14, 15]. However, the Wald-type confidence intervals can perform poorly for small to moderate sample sizes due to poor estimation of the standard deviation of the estimator or if the sampling distribution of the estimator is strongly skewed.

This paper considers an alternative method known as the profile likelihood method to construct confidence intervals for the parameters α and β of the NHPP with linear intensity function (1) based the occurrence of events data. It is assumed that the overall time interval $(0, T)$ is divided in N measurement subintervals $\left(\frac{(k-1)T}{N}, \frac{kT}{N}\right], 1 \leq k \leq N$ and the number of occurrences of the event of interest is observed in each.

The profile likelihood intervals use all the information encoded in the likelihood function concerning the parameters and are likely to be more robust in small samples. The researchers in [9] constructed the profile likelihood confidence intervals for the parameters of item response models and used simulation to demonstrate that they consistently perform better than Wald confidence intervals. In capture-recapture analysis, it is known that the small sample distributions of the parameter estimators are markedly skewed and thus deviates from normality. As a result, the Wald confidence intervals are frequently not viable. Therefore, Evans, M. A. et al. [12] and Gimenez, O. et al. [10] proposed the profile likelihood approach to the construction of confidence intervals for the size of a closed population (N) from a capture-recapture data. Constrained optimization was used by Reich, G. et al. [11] to develop a simple and efficient approach to computing profile likelihood confidence intervals and compared their approach to other particular types of confidence intervals.

The remainder of this paper is organized as follows: Section 2 presents the formulation of the mathematical model of the nonhomogeneous Poisson process with linear rate, the procedures for constructing profile likelihood and Wald confidence intervals are described in Sections 3 and 4, respectively, Section 5 details the simulation study that was designed to compare the above described procedures of constructing confidence intervals, and Section 6 is the conclusion.

2. Mathematical Formulation of the Model

2.1. Nonhomogeneous Poisson Process with Linear Rate

Let $N(t)$ be the number of events that occur during the time interval $(0, t]$. Consider an integrable function $\lambda: [0, \infty) \to [0, \infty)$. A counting process $\{N(t), t \geq 0\}$ is called a nonhomogeneous process (NHPP) with rate (or intensity) $\lambda(t)$ if:

1) $N(0) = 0$
2) For each $t > 0, N(t)$ has a Poisson distribution with mean $m(t) = \int_0^t \lambda(s)ds$
3) For each $0 \leq t_1 < t_2 < \cdots < t_m$, $N(t_1), N(t_2) - N(t_1), \cdots, N(t_m) - N(t_{m-1})$ are independent random variables.

For a NHPP with intensity $\lambda(t)$, the number of occurrences of an event in any interval is a Poisson random variable. Thus for each $0 \leq s < t$, $N(t) - N(s)$ is Poisson distributed with mean $m(t) - m(s) = \int_s^t \lambda(u)du$.

Suppose that we have a NHPP with the rate function $\lambda(t) = \alpha + \beta t$ over the interval $[0, T]$ as in (1). The number of occurrences of the event of interest is observed in each of the N subintervals $\left(\frac{(k-1)T}{N}, \frac{kT}{N}\right], 1 \leq k \leq N$. Let Y_k denote the number occurrences of the event in the k^{th} subinterval. Then Y_k are independent Poisson random variables with mean

$$\lambda_k = \frac{T}{N}(\alpha + \beta x_k), \tag{2}$$

where

$$x_k = \left(k - \frac{1}{2}\right)\frac{T}{N}, 1 \leq k \leq N \tag{3}$$

2.2. The Maxim Likelihood Estimation

Let $\underline{y} = (y_1, y_2, \cdots, y_N)$ be a realization of the count vector $\underline{Y} = (Y_1, Y_2, \cdots, y_N)$ for the N subintervals. Since $Y_k's$ are independent, the likelihood function (joint density) is the product of the densities of $Y_k's$:

$$L(\alpha, \beta) = \prod_{k=1}^{N} \frac{e^{-\lambda_k}(\lambda_k)^{y_k}}{y_k!}$$

$$= e^{-\sum_{k=1}^{N}\lambda_k} \prod_{k=1}^{N} \frac{(\lambda_k)^{y_k}}{y_k!},$$

and the log-likelihood function is given as

$$l(\alpha, \beta) = logL(\alpha, \beta) = -\sum_{k=1}^{N}\lambda_k + \sum_{k=1}^{N} y_k log\lambda_k - \sum_{k=1}^{N} log(y_k!)$$

$$= \alpha T - \beta\frac{T^2}{2} + \sum_{k=1}^{N} y_k log\left((\alpha + \beta x_k)\frac{T}{N}\right) - \sum_{k=1}^{N} log(y_k!) \tag{4}$$

Differentiating $l(\alpha, \beta)$ partially with respect to α and β, and setting the resulting derivatives to zero yields the following two equations:

$$\sum_{k=1}^{N} \frac{y_k}{\alpha + \beta x_k} = T \tag{5}$$

and

$$\sum_{k=1}^{N} \frac{x_k y_k}{\alpha + \beta x_k} = \frac{T^2}{2} \tag{6}$$

The maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ of α and β are then obtained by numerically solving equations (5) and (6) simultaneously.

3. Profile Likelihood Confidence Intervals

The principle of profile likelihood method for constructing confidence intervals is described as follows: Suppose $\hat{\underline{\theta}}$ is the MLE of the vector of model parameters $\underline{\theta} \in \Theta \subseteq \mathfrak{R}^k$ and $l(\underline{\theta})$ be the log-likelihood function. Let θ_j be considered as a single parameter of interest and the others as nuisance parameters. The profile log-likelihood function of θ_j, denoted by $l_p(\theta_j)$, is obtained by holding θ_j fixed and maximizing $l(\underline{\theta})$ over the other $k-1$ parameters. That is

$$l_p(\theta_j) = \max_{\theta_j\, fixed} l(\underline{\theta}) \tag{7}$$

Consequently, the relative profile log-likelihood function of the parameter θ_j, denoted by $r_p(\theta_j)$, is defined as

$$r_p(\theta_j) = l_p(\theta_j) - l(\underline{\hat{\theta}}). \tag{8}$$

If the number of unknown parameters is small in comparison with the number of independent observations, then the relative profile log-likelihood function has properties similar to those of a one parameter relative log-likelihood function and satisfactory results will be obtained [13]. Inferences concerning θ_j can be made using the relative profile log-likelihood function. The $100(1-\gamma)\%$ profile likelihood confidence interval for θ_j is the set of parameter values given by $\{\theta_j: r_p(\theta_j) \geq -\frac{1}{2}\chi^2_{\frac{\gamma}{2},1}\}$, where $\chi^2_{\frac{\gamma}{2},1}$represents the $\left(1-\frac{\gamma}{2}\right)$ quantile of the chi-square distribution with 1 degree of freedom.

The general profile likelihood technique described above can be used to construct confidence intervals for the parameters α and β of the NHPP with linear rate. In this case the parameter space $\underline{\theta} = (\alpha, \beta)$ is two dimensional and on the basis of the log-likelihood function in (4) the respective relative profile log-likelihood functions $r_p(\alpha)$ and $r_p(\beta)$ are given by

$$r_p(\alpha) = l\left(\alpha, \hat{\beta}(\alpha)\right) - l(\hat{\alpha}, \hat{\beta}) \tag{9}$$

and

$$r_p(\beta) = l(\hat{\alpha}(\beta), \beta) - l(\hat{\alpha}, \hat{\beta}) \tag{10}$$

Here $\hat{\beta}(\alpha)$ is the MLE of β with respect to $l(\alpha, \beta)$for fixed α and similarly, $\hat{\alpha}(\beta)$ is the MLE of α with respect to $l(\alpha, \beta)$for fixed β. Calculation of the relative profile log-likelihood functions $r_p(\alpha)$ and $r_p(\beta)$ can be done numerically and the $100(1-\gamma)\%$ profile confidence intervals can constructed.

4. Wald Confidence Intervals

The Wald method can be applied in construction of confidence intervals in multiparameter models where $\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_k)$ is the vector of unknown the parameters. Let $\hat{\theta}_j$ be the MLE of θ_j, $j = 1,2,\cdots,k$. Then by the asymptotic normality property of MLE, the sampling distribution of $\frac{\hat{\theta}_j - \theta_j}{se(\hat{\theta}_j)}$ is approximately standard normal, where the standard error $se(\hat{\theta}_j)$ is defined as the square root of the j^{th} diagonal entry of the inverse observed Fisher information matrix. That's $se(\hat{\theta}_j) = \sqrt{\left[I(\underline{\hat{\theta}})^{-1}\right]_{jj}}$. The $100(1-\gamma)\%$ Wald confidence interval of a single parameter θ_j is given by $\hat{\theta}_j \pm z_{\frac{\gamma}{2}} se(\hat{\theta}_j)$.

The observed Fisher Information matrix $I(\hat{\alpha}, \hat{\beta})$ for the log-likelihood function in (4) is a square matrix of order 2 containing negative second partial derivatives of the log-likelihood function $l(\alpha, \beta)$:

$$I(\hat{\alpha}, \hat{\beta}) = \begin{bmatrix} -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 l}{\partial \alpha \partial \beta} & -\frac{\partial^2 l}{\partial \beta^2} \end{bmatrix}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} \tag{11}$$

where

$$\frac{\partial^2 l}{\partial \alpha^2} = -\sum_{k=1}^{N} \frac{y_k}{(\alpha+\beta x_k)^2} \tag{12}$$

$$\frac{\partial^2 l}{\partial \beta^2} = -\sum_{k=1}^{N} \frac{x_k^2 y_k}{(\alpha+\beta x_k)^2} \tag{13}$$

and

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = -\sum_{k=1}^{N} \frac{x_k y_k}{(\alpha+\beta x_k)^2} \tag{14}$$

The observed Fisher information matrix can be inverted to obtain a local estimate of the asymptotic variance covariance matrix of the MLE $(\hat{\alpha}, \hat{\beta})$ as

$$I(\hat{\alpha}, \hat{\beta})^{-1} = \begin{bmatrix} \hat{\sigma}_{11}(\hat{\alpha}, \hat{\beta}) & \hat{\sigma}_{12}(\hat{\alpha}, \hat{\beta}) \\ \hat{\sigma}_{21}(\hat{\alpha}, \hat{\beta}) & \hat{\sigma}_{22}(\hat{\alpha}, \hat{\beta}) \end{bmatrix} \tag{15}$$

where $\hat{\sigma}_{ij}(\hat{\alpha}, \hat{\beta}) = \left[I(\hat{\alpha}, \hat{\beta})^{-1}\right]_{ij}$.

Consequently, a $100\left(1-\frac{\gamma}{2}\right)\%$ Wald confidence interval for α can be constructed as

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}_{11}(\hat{\alpha}, \hat{\beta})} \tag{16}$$

Similarly, the $100\left(1-\frac{\gamma}{2}\right)\%$ Wald confidence interval for β can be constructed as

$$\hat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}_{22}(\hat{\alpha}, \hat{\beta})} \tag{17}$$

5. Simulation Study

The performances of the two interval estimation methods, namely, Wald method and Profile likelihood method, were investigated on the basis of data simulated from a NHPP with linear rate under different cases determined by the values assigned to the four parameters N, T, α and β. An R code was developed to generate mutually independent Poisson random variables Y_k with means λ_k in (1) and to construct the Wald and Profile likelihood intervals for the two unknown parameters α and β. In the first case a set of data $y_k, k = 1,2,\cdots.N$ was simulated when $N = 20, T = 300, \alpha = 1, \beta = 0.5$ and graphs of the relative profile likelihood function and the normal approximation were plotted for each of the two parameters as shown in figure 1. For both parameters the graph of the normal approximation in green dotted line offers a good approximation to the respective profile relative likelihood function indicating a good performance of the Wald confidence intervals. However, for the parameter α the slight departure of the graph of normal approximation from that of the relative profile likelihood function, the Wald confidence intervals will include implausible values of the parameter. Next, five cases determined by the values of N were considered and for each value of N 500 datasets were simulated. The Profile likelihood and Wald 95% confidence intervals were constructed for each dataset and their widths calculated. The estimated coverage probability for each of the two interval methods was the proportion of the 500 confidence intervals which include the true parameter value. These estimated coverage probability of the two confidence interval types and the summary statistics (minimum, mean, maximum, standard deviation) of the widths of the 500 confidence intervals for each case are reported in table 1. The results in table 1 show that for all the cases considered the Profile likelihood confidence intervals have smaller minimum, mean and maximum interval widths as compared to those of Wald confidence interval, but it can be observed that as N increases the mean interval width for the Profile likelihood confidence intervals approaches that of the Wald confidence intervals. Also, it was observed that for the five values of N used the respective numbers of 95% confidence intervals for the parameter α with negative lower limits out of the 500 Wald confidence intervals were 189, 155, 143, 139 and 142.

Table 1. *Estimated coverage probabilities and summary statistics of widths of the 95% Profile Likelihood and Wald confidence intervals for the parameters* α *and* β*on the basis of 500 replications when* $T = 300, \alpha = 1, \beta = 0.5$ *for each*$N = 20, 50, 100, 200, 500$*.*

N			Min	Mean	Max	std dev	Cp
20	Profile	α	0.365	1.602	1.891	0.248	0.948
		β	0.01202	0.01726	0.01808	0.0007	0.940
	Wald	α	1.557	1.744	1.954	0.059	0.946
		β	0.01681	0.01742	0.01805	0.0002	0.936
50	Profile	α	0.505	1.530	1.800	0.203	0.962
		β	0.01191	0.01675	0.01780	0.0008	0.930
	Wald	α	1.397	1.623	1.835	0.084	0.954
		β	0.01589	0.01692	0.01788	0.0003	0.932
100	Profile	α	0.579	1.489	1.793	0.199	0.952
		β	0.00928	0.01668	0.01797	0.0008	0.942
	Wald	α	1.186	1.579	1.857	0.108	0.946
		β	0.01499	0.01669	0.01792	0.0005	0.958
200	Profile	α	0.130	1.489	1.800	0. 216	0.942
		β	0.00687	0.01659	0.01783	0.0008	0.954
	Wald	α	1.054	1.575	1.868	0.124	0.944
		β	0.01492	0.01669	0.01792	0.0005	0.958
500	Profile	α	0.363	1.479	1.793	0.207	0.944
		β	0.00393	0.01645	0.01776	0.00110	0.953
	Wald	α	0.960	1.565	1.565	1.826	0.944
		β	0.01431	0.01666	0.01784	0.0005	0.974

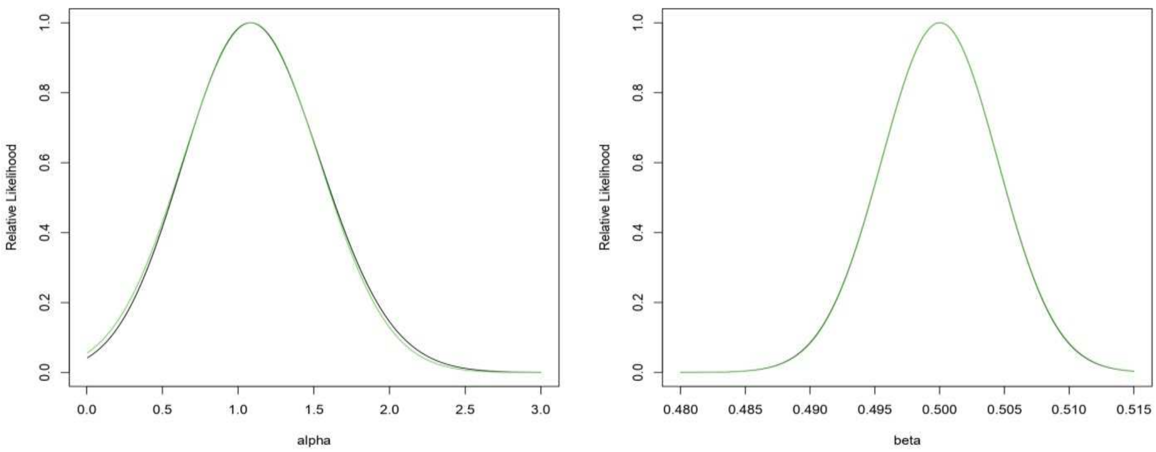

Figure 1. *The graphs of Profile relative likelihood function (continuous black line) and the Normal approximation (dotted green line) of the parameters* α *and* β *when* $N = 200, T = 300, \alpha = 1, \beta = 0.5$*.*

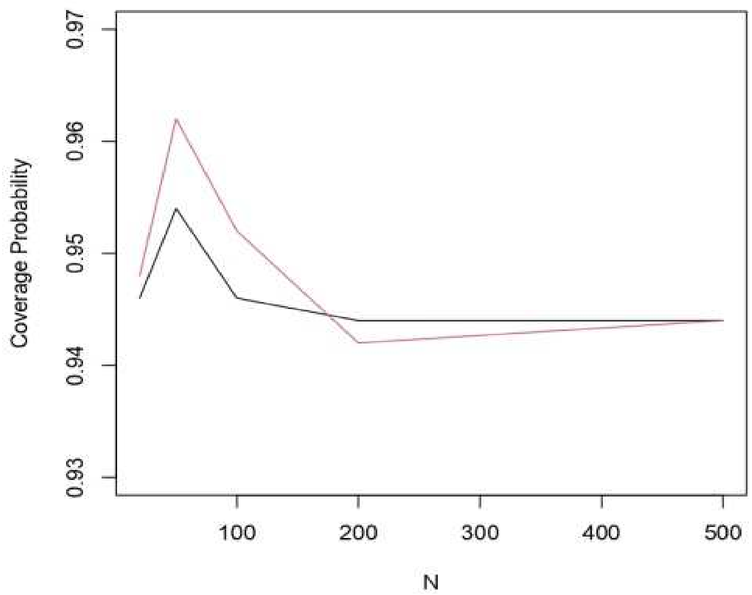

Figure 2. *Plots of coverage probabilities for Profile likelihood (dark line) and Wald (colored line) 95% confidence intervals for the parameter* α *over the values of* N.

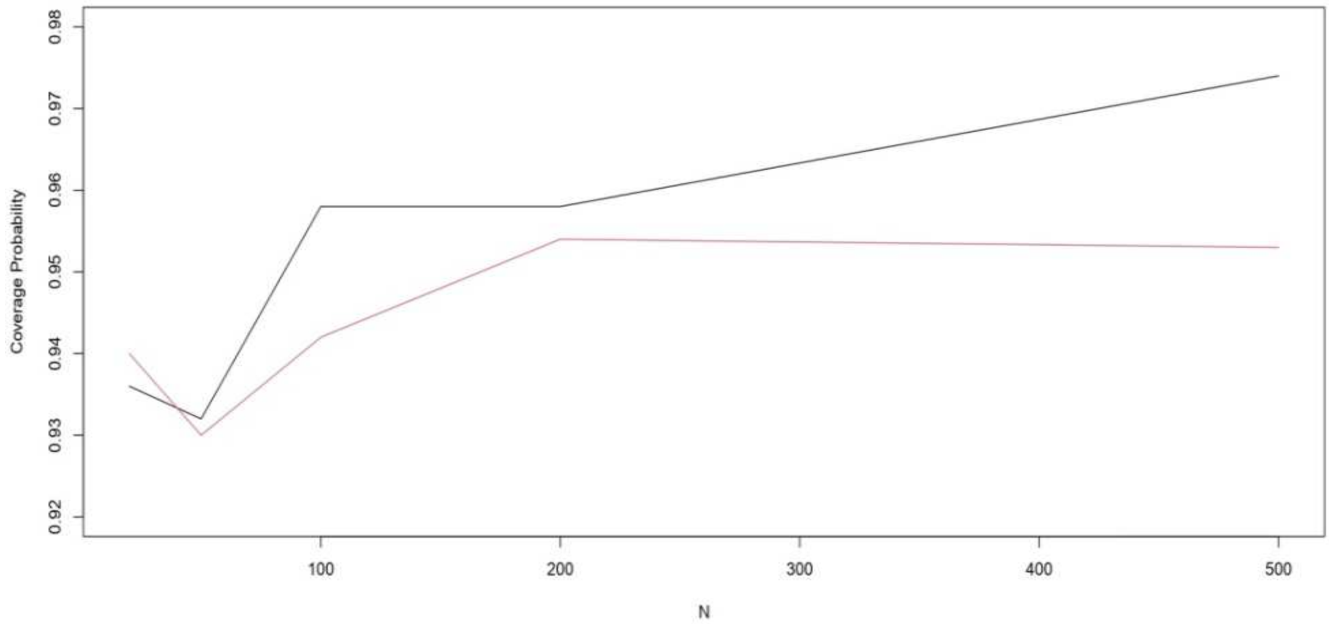

Figure 3. *Plots of coverage probabilities for Profile likelihood (dark line) and Wald (colored line) 95% confidence intervals for the parameter* β *over the values of* N.

The values of the estimated coverage probabilities of confidence intervals constructed by the two interval estimation methods for the parameters α and β are reasonably close to the nominal value 0.95 and their variations over the values of N are shown in figures 2 and 3.

Figures 4 and 5 are plots of the widths of the 500 Profile likelihood and Wald confidence intervals for the parameters α and β, respectively. In figure 4, interval widths for the Profile likelihood confidence intervals for α show upward spikes which are short and uniform and long variable downward spikes, while upward and downward spikes for the widths of the corresponding Wald confidence intervals are short and uniform. On average the mean interval widths for the two method do not differ much but the long downward spikes in the first graph indicates that the method of Profile likelihood is likely to produce shorter confidence intervals than the Wald method.

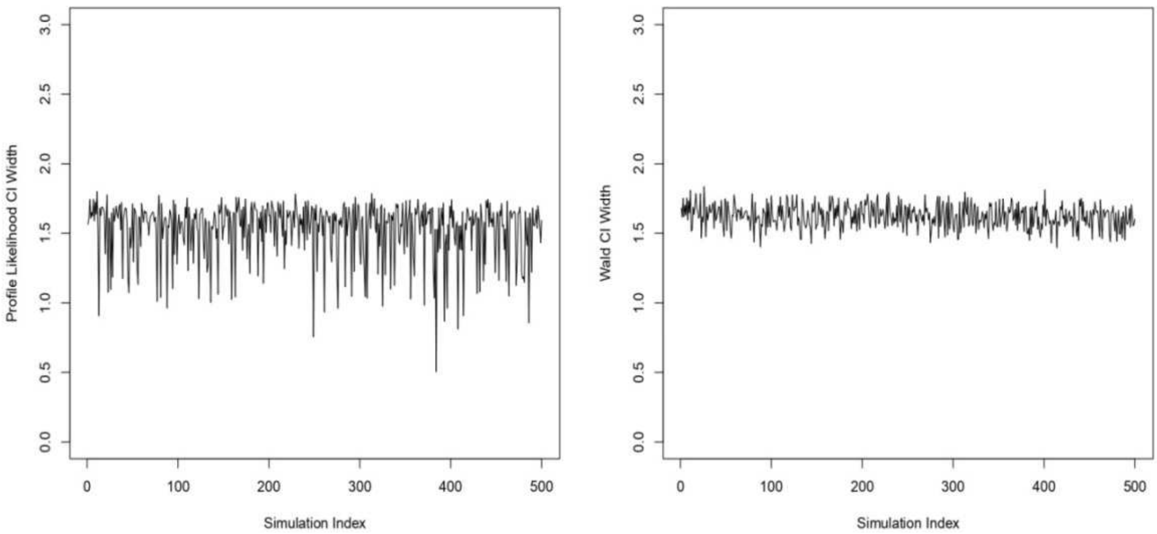

Figure 4. *Widths of 95% Profile likelihood and Wald confidence intervals for* α *when* $N = 50, T = 300, \alpha = 1, \beta = 0.5$ *corresponding to 500 replications.*

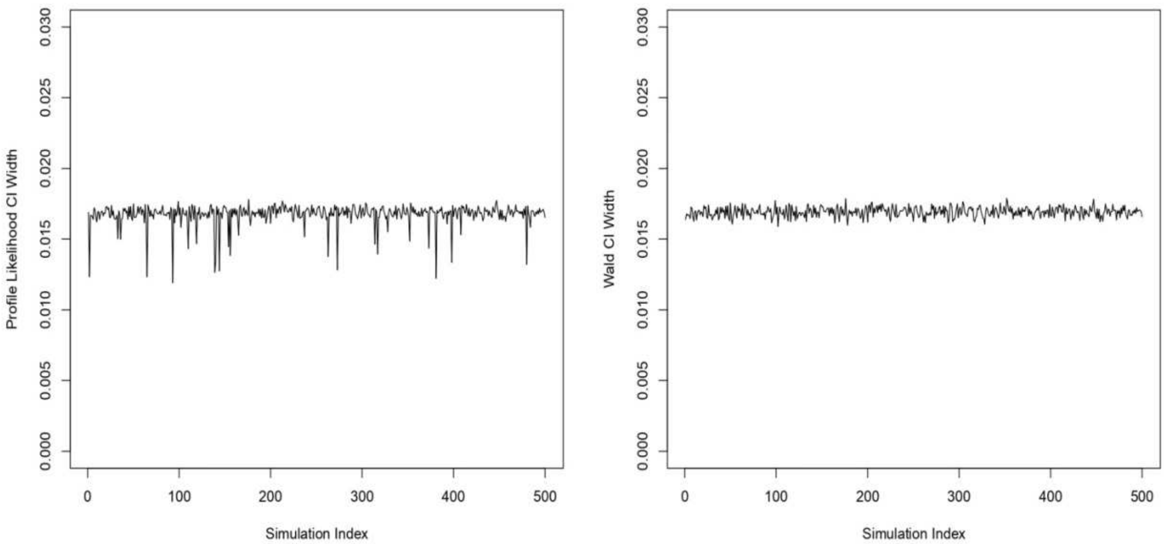

Figure 5. *Widths of 95% Profile likelihood and Wald confidence intervals for* β *when* $N = 50, T = 300, \alpha = 1, \beta = 0.5$ *corresponding to 500 replications.*

In figure 5, the spikes for the interval widths for the confidence intervals produced by the two method display a similar behavior as in figure 4, however, the long downward spikes in the first graph are fewer and the degree of uniformity is higher indicating increased precision in the two interval estimation methods. The values of standard deviation in the second last column of table 1 quantifies the precision the Profile likelihood and Wald methods in all the cases studied and explain the nature of the spikes displayed in figures 4 and 5.

6. Conclusion

This article considered Profile likelihood method as an alternative method for constructing approximate confidence intervals for the parameters of NHPP with linear rate. On the basis of simulated data the Profile likelihood confidence intervals were compared with the large sample Wald confidence intervals in terms of interval width and coverage probability. The results of the simulation study displayed in terms of a table and graphs above show that the Wald confidence intervals provide good approximation to the Profile likelihood intervals in terms the two efficiency measures used. However, in all the cases considered the Profile likelihood method produced shorter confidence intervals for both parameters as compared to the Wald method, which produced some confidence intervals with negative lower limits for the non-negative parameter α. These Wald confidence intervals with negative lower limits were observed to be many for small values of N, where normality approximation is not expected to be good. Therefore, it is worth concluding that the profile likelihood method is superior to the Wald method since it yields confidence intervals containing plausible values of the two parameters. Even though these results have been demonstrated for the cases considered in this article, other cases determined by the values of T, α and β could be studied to increase the scope of the comparison of these two interval estimation methods.

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