

# Properties and Construction Method for Symmetric Balanced Incomplete Block Design with $\lambda=1$

Troon John Benedict<sup>1</sup>, Onyango Fredrick<sup>2</sup>, Karanjah Anthony<sup>3</sup>

<sup>1</sup>Department of Mathematics and Physical Sciences, Maasai Mara University, Narok, Kenya

<sup>2</sup>Department of Mathematics and Actuarial Science, Maseno University, Luanda, Kenya

<sup>3</sup>Department of Mathematics, Multimedia University, Nairobi, Kenya

## Email address:

troon@mmarau.ac.ke (Troon John Benedict), fonyango@maseno.ac.ke (Onyango Fredrick), akaranjah@mmu.ac.ke (Karanjah Anthony)

## To cite this article:

Troon John Benedict, Onyango Fredrick, Karanjah Anthony. Properties and Construction Method for Symmetric Balanced Incomplete Block Design with  $\lambda=1$ . *American Journal of Theoretical and Applied Statistics*. Vol. 12, No. 1, 2023, pp. 13-17. doi: 10.11648/j.ajtas.20231201.12

Received: April 13, 2023; Accepted: April 27, 2023; Published: May 10, 2023

**Abstract:** Symmetric Balanced Incomplete Block Designs with  $\lambda=1$  is a common class of BIBDs which are mostly used in incomplete experimental block design set up because of their simplicity in set up and also in analysis. Over the years since development of the BIBDs by Yates in the year 1939. A number of research has been done on the design to establish properties of the design and also to determine the construction methods of the design. In terms of properties, the studies have only been able to establish necessary but not sufficient conditions for the existence of the design. For the symmetric BIBDs the studies have also determined the non-existence properties for such designs. However, the sufficient existence property for the design have not been established. In terms of construction, the studies have been able to derive several construction methods for BIBDs. However, these methods have been determined not to be adequate in constructing all the BIBDs which still leave the existence of some BIBDs as unknown. For symmetric BIBDs with  $\lambda=1$  which are also known as projective planes, the studies have not been able to establish the sufficient properties for existence of this class of BIBDs just like the other classes of symmetric BIBDs. Therefore, this give room for investigating other properties of this class of BIBDs. The present study therefore, aimed at deriving the properties of the design from the known properties of BIBDs and also using the properties to determine the construction technique that would be suitable used in constructing this class of BIBDs. The study used the known properties of symmetric BIBDs to derive new properties of symmetric BIBDs, then restricted it to the case of  $\lambda=1$ . Which aided in derivation of new properties of the design and also the construction method. The study was able to derive three new properties for this class of BIBD and it was also able to show that the class of BIBD would be best constructed using PG(2,S).

**Keywords:** Symmetric BIBD, Projection Geometry, Perfect Odd Square, Galos Field

## 1. Introduction

A Balanced Incomplete Block Design (BIBD) is an arrangement of  $v$  treatments into  $b$  blocks each of size  $k$  such that each treatment is replicated  $r$  times in the design while each pair of treatments occur together  $\lambda$  times in the entire design [6, 9, 16]. The design is said to be symmetric when  $b = v$  and  $r = k$ . In order for a BIBD to exist, it is always required that the parameters of the BIBD must satisfy a number of necessary conditions [1, 2, 4, 7, 9, 11, 12, 16]. First, the total number of plots in the design ( $bk$ ) is always known to be equivalent to the number of treatments  $v$  times the number of replication  $r$ . Thus,

$$bk = rv \quad (1)$$

This property can always be derived by considering the total number of blocks  $b$  in the design, to be given by;

$$\begin{aligned} b &= r \frac{\binom{v}{k}}{\binom{v-1}{k-1}} \\ &= r \frac{v!}{k!(v-k)!} \div \frac{(v-1)!}{(k-1)!(v-k)!} \\ &= r \frac{v!}{k!(v-k)!} \times \frac{(k-1)!(v-k)!}{(v-1)!} \\ &= r \frac{v(v-1)!}{k(k-1)!(v-k)!} \times \frac{(k-1)!(v-k)!}{(v-1)!} \\ &= \frac{rv}{k} \\ bk &= rv \end{aligned}$$

The second property, tells us that the total number of treatments that occur together with a particular treatment is in the design  $\lambda(v-1)$  must always be equal to the number of replications  $r$  times the number of plots per block less 1 plot  $(k-1)$ . Thus,

$$\lambda(v-1) = r(k-1) \quad (2)$$

This property also can be derived from the total number of block  $b$  in the design, that is

$$\begin{aligned} b &= \lambda \frac{\binom{v}{k}}{\binom{v-2}{k-2}} \\ &= \lambda \frac{v!}{k!(v-k)!} \div \frac{(v-2)!}{(k-2)!(v-k)!} \\ &= \lambda \frac{v!}{k!(v-k)!} \times \frac{(k-2)!(v-k)!}{(v-2)!} \\ &= \lambda \frac{v(v-1)(v-2)!}{k(k-1)(k-2)!(v-k)!} \times \frac{(k-2)!(v-k)!}{(v-2)!} \\ &= \lambda \frac{v(v-1)}{k(k-1)} \end{aligned}$$

But it is known that  $b = \frac{rv}{k}$ . Therefore, it can be shown that

$$\begin{aligned} \frac{rv}{k} &= \lambda \frac{v(v-1)}{k(k-1)} \\ r &= \lambda \frac{v(v-1)k}{k(k-1)v} \\ &= \lambda \frac{v-1}{k-1} \\ \lambda(v-1) &= r(k-1) \end{aligned}$$

The properties in Equations 1 and 2, are known as the fundamental or basic properties of a BIBD which must always be satisfied for the design to exist. However, over the years, other properties that applies for both symmetric and non-symmetric BIBD have been ascertained as given in Equations 3 and 4. The property in Equation 3 requires that for a BIBD to exist then the number of times each treatment is replicated  $r$  must be greater than the number of times the treatment occur together with other treatments.

$$r > \lambda \quad (3)$$

Fisher's also introduced a property that is required for a BIBD to exist which given as

$$b \geq v \quad (4)$$

The property in Equation 4 is known as the Fisher's Inequality. Past studies have determined that always when properties in Equations 1 and 2 are satisfied then mostly the properties in Equations 3 and 4 are also satisfied. One of the major breakthrough on BIBDs was the establishment of the non-existence property for symmetric BIBDs. Consider an incidence matrix  $T$

$$T = \begin{bmatrix} v_{11} & v_{12} & v_{13} & \dots & v_{1b} \\ v_{21} & v_{22} & v_{23} & \dots & v_{2b} \\ v_{31} & v_{32} & v_{33} & \dots & v_{3b} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{v1} & v_{v2} & v_{v3} & \dots & v_{vb} \end{bmatrix}$$

Such that

$$v_{ij} = \begin{cases} 1 & \text{if treatment } v_i \text{ is in block } b_j \\ 0 & \text{if treatment } v_i \text{ is not in block } b_j \end{cases}$$

The matrix  $T$  is called an incidence matrix with  $b$  columns and  $v$  rows. In each column there are a total of  $k$  ones and  $v-k$  zeros. On the other hand, in each row, there are  $r$  ones and  $b-r$  zeros. Given an incidence matrix  $T$  of a BIBD;

$$\det(TT') = (r-\lambda)^{v-1}k^2 \quad (5)$$

If the incidence matrix  $T$  is of a symmetric BIBD, then it follows that;

$$\det(T) = (r-\lambda)^{\frac{v-1}{2}}k \quad (6)$$

Therefore, according to Akral et al; Greig and Rees; and Hsiao-Lih et al, [1, 7, 10], for a non-existence of a symmetric BIBD, then if  $v$  is even, then  $r-\lambda$  is not a perfect square. According to Bruck-Ryser-Chowla theorem, for a symmetric BIBD with parameters  $v=b$ ,  $k=r$  and  $\lambda$ . For a symmetric BIBD with number of treatments  $v$  which is odd. In order for the design not to exist then the equation

$$x^2 = (r-\lambda)y^2 + (-1)^{\frac{v-1}{2}}\lambda z^2 \quad (7)$$

should not have a solution in integers  $x, y, z$  not all simultaneously zero [6]. Similar to the properties of BIBDs, much stride has been made in construction of BIBDs. Several construction methods of BIBD have been discovered over the years which include; Use of projection geometry, Euclidean geometric, Cyclic difference sets, Repeated difference sets, Latin square technique, Linear Integer Algorithm, R ibd package, using existing BIBD designs among others [1-5, 8-11, 13-15, 17-20]. Looking into construction of BIBD using projection geometry. According to Bose; and Greig and Rees [4, 7], consider a ordered set  $N+1$  i.e  $y_1, y_2, y_3, \dots, y_{N+1}$  such that  $y_1 \in GF(S)$  with the  $y_i$ s not all simultaneously zeros. We can use finite projection geometry of  $N$  dimension to construct a BIBD as follows. Based on the  $N$  dimension projection geometry, the total number of points ( $n_p$ ) in the design is given by the formula;

$$n_p = \frac{s^{N+1}-1}{s-1} \quad (8)$$

where  $s = p^n$

The entire set of points of the projection geometry satisfy a set of  $N-m$  independent sets of linear homogeneous equations known as an  $m-flat$  which is given as

$$\begin{aligned} k_{10}y_0 + k_{11}y_1 + k_{12}y_2 + \dots + k_{1N}y_N &= 0, \\ k_{20}y_0 + k_{21}y_1 + k_{22}y_2 + \dots + k_{2N}y_N &= 0, \\ &\vdots \\ k_{N-m,0}y_0 + k_{N-m,1}y_1 + k_{N-m,2}y_2 + \dots + k_{N-m,N}y_N &= 0, \end{aligned} \quad (9)$$

Based on the set of Equations in 9, a  $0-flat$  is a point, a  $1-flat$  is a line and a  $2-flat$  is a plane. The total number of  $m-flats$  in the projection geometry is denoted by  $\phi(N, m, s)$  and is given by;

$$\phi(N, m, s) = \frac{(s^{N+1}-1)(s^N-1)\dots(s^{N-m+1}-1)}{(s^{m+1}-1)(s^m-1)\dots(s-1)} \quad (10)$$

Based on the projection geometry  $PG(N, p^n)$  and the  $m$ -flat we can construct a BIBD as follows; We let every point in the projection geometry to correspond to a treatment such the total number of treatments in the design is given by the formula;

$$v = \phi(N, 0, s) = \frac{s^{N+1}-1}{s-1} \quad (11)$$

We let the blocks corresponds to the  $m$ -flats with every point in each  $m$ -flat corresponding to a treatment in the block. Therefore, the total number of blocks in the design will be given by the formula;

$$b = \phi(N, m, s) = \frac{(s^{N+1}-1)(s^N-1)\dots(s^{N-m+1}-1)}{(s^{m+1}-1)(s^m-1)\dots(s-1)}$$

The number of points in each block which corresponds to the number of plots per block is therefore given by the formula:

$$k = \phi(m, 0, s) = \frac{s^{m+1}-1}{s-1} \quad (12)$$

In such a design the number of times each treatment will be repeated ( $r$ ) is given as  $r = \phi(N-1, m-1, s)$  while the number of times a pair of treatment occur together ( $\lambda$ ) is given as  $\lambda = \phi(N-2, m-2, s)$ . If  $N$  and  $m$  are constant, then the BIBD designs obtained for different values of  $s$  are said to belong to the same series  $P_N^m$ . The total number of lines is also given by the same formula in Equation (); Using the number of points and lines from the projection geometry, we can arrange the points (treatments ( $v$ )) into lines (blocks( $b$ )) of size  $k = s + 1$  such that each point (treatment) occurs  $r = s + 1$  times in total. Therefore, a  $PG(2, S)$  can be used to construct BIBD with parameters;

$$v = S^2 + S + 1 = b \quad (13)$$

$$k = S + 1 = r \quad (14)$$

## 2. Additional Properties for Symmetric BIBDs

**Theorem 1.** Consider a symmetric BIBD with parameters  $(v = b, r = k, \lambda)$ . For such a BIBD to exist, then  $1 + 4\lambda(v-1)$  must be a perfect odd square.

*Proof.* From property in Equation 2, for a symmetric BIBD  $r = k$  and  $b = v$ , we get

$$\lambda(v-1) = k(k-1)$$

Solving, we get

$$\begin{aligned} k^2 - k &= \lambda(v-1) \\ k^2 - k - \lambda(v-1) &= 0 \end{aligned}$$

which implies that

$$k = \frac{1 \pm \sqrt{1 + 4\lambda(v-1)}}{2} \quad (15)$$

since,  $k$  cannot be a negative number then we only use the positive solution for Equation 15. that is

$$k = \frac{1 + \sqrt{1 + 4\lambda(v-1)}}{2} \quad (16)$$

Now given that  $k$  must be an integer, then Equation 16 can only satisfy this condition if  $1 + 4\lambda(v-1)$  is a perfect odd square. Hence the proof.

**Theorem 2.** Consider a symmetric BIBD with parameters  $(v, k, \lambda)$ . For such a BIBD to exist, then  $\lambda(v-1)$  must be an even number.

*Proof.* From Theorem Theorem 1,  $1 + 4\lambda(v-1)$  is a perfect odd square. That is  $1 + 4\lambda(v-1)$  can be written as  $4S^2 + 4S + 1$  and thus, dividing by 8 on both sides we get

$$\begin{aligned} \frac{4\lambda(v-1)}{8} &= \frac{4S^2 + 4S}{8} \\ \frac{\lambda(v-1)}{2} &= \frac{S(S+1)}{2} \end{aligned}$$

Given that  $S(S+1)$  is divisible by 2 therefore its even, it also follows that  $\lambda(v-1)$  is also divisible by 2 hence also even.

*End of proof*

**Corollary 1.** Consider a symmetric BIBD with parameters  $(v, k, \lambda)$ . For such a BIBD to exist, then if  $\lambda$  is odd then  $v$  must be odd and if  $\lambda$  is even then  $v$  can be even or odd.

*Proof.* Now given the  $\lambda(v-1)$  is even, then it follows that if  $\lambda$  is odd then  $v-1$  must be even which can only be true is  $v$  is odd. On the other hand, if  $\lambda$  is even then  $v-1$  can be odd or even.

## 3. Analysis of $\lambda = 1$ for the Case of Corollary Corollary 1

The study analyzed the corollary Corollary 1 for the case of  $\lambda = 1$  in order to see whether the corollary holds and if it can be used to generate symmetric BIBDs for  $\lambda = 1$ . When  $\lambda = 1$  then the Equation 16 give rise into;

$$k = \frac{1 + \sqrt{4v-3}}{2} \quad (17)$$

As a result, for a BIBD to exist the the following must be satisfied.

$(4v-3)$  must be a perfect odd square

$(v-1)$  is even

For incidence matrix  $T$ , the  $\det(T)$  is an integer. Thus,

$$\det(T) = (r - \lambda)^{\frac{v-1}{2}} k$$

However, if  $\lambda = 1$  and the BIBD is symmetric then it follows that

$$v-1 = k(k-1) \quad (18)$$

Hence, it follows that

$$\det(T) = (r - \lambda)^{\frac{k(k-1)}{2}} k \quad (19)$$

The Equation 19 give rise into an integer under all values of  $k$ .

#### 4. Construction of Symmetric BIBD with $\lambda = 1$

Theorem 3. Consider a symmetric BIBD with parameters  $v = b, r = k, \lambda = 1$ . This class of BIBD can be constructed using projection geometry  $PG(2, S)$ .

*Proof.* From Equation 17 it follows that  $(4v - 3)$  is a perfect odd square. That is  $4v - 3$  can be written as;

$$\begin{aligned} 4v - 3 &= (2S + 1)^2, \forall S = 1, 2, 3, \dots, n \\ 4v &= 4S^2 + 4S + 4 \\ v &= S^2 + S + 1 = b \end{aligned}$$

This refers to the total number of points in a  $PG(2, S)$ . Similarly, it follows from Equation 17, that  $2k - 1$  is a odd number, since  $4v - 3$  is a perfect odd square and thus  $\sqrt{(4v - 3)}$  is odd. Thus,  $2k - 1$  can be written as;

$$\begin{aligned} 2k - 1 &= 2S + 1 \\ 2k &= 2S + 2 \\ k &= S + 1 = r \end{aligned}$$

Which is the total number of replications and plot size in a BIBD constructed from  $PG(2, S)$ , this shows that for a BIBD with  $\lambda = 1$ , the designs can be constructed with a  $PG(2, S)$ .

End of proof.

Now using the parameter relation with  $S$ , the list of symmetric BIBD parameters with  $v = b, k = r, \lambda = 1$  listed in Table 1 could be constructed using  $PG(2, S)$ ;

**Table 1.** List of Symmetric BIBD with  $\lambda = 1$  and  $v < 600$  for  $S = 1, 2, 3, \dots, 23$ .

S	v	B	r	k
1	3	3	2	2
2	7	7	3	3
3	13	13	4	4
4	21	21	5	5
5	31	31	6	6
6	43	43	7	7
7	57	57	8	8
8	73	73	9	9
9	91	91	10	10
10	111	111	11	11
11	133	133	12	12
12	157	157	13	13
13	183	183	14	14
14	211	211	15	15
15	241	241	16	16
16	273	273	17	17
17	307	307	18	18
18	343	343	19	19
19	381	381	20	20
20	421	421	21	21
21	463	463	22	22
22	507	507	23	23
23	553	553	24	24

However, for a  $PG(2, S)$   $S$  must be a member of Galos field meaning that  $S$  must either be a prime number or a power of a prime number. Therefore, this show that the list of symmetric BIBDs with  $\lambda = 1$  that can be constructed using  $PG(2, S)$  is as show in table 2.

**Table 2.** List of Symmetric BIBD with  $\lambda = 1$  and  $v < 600$  that can be constructed from  $PG(2, S)$ .

S	V	b	r	K
1	3	3	2	2
2	7	7	3	3
3	13	13	4	4
4	21	21	5	5
5	31	31	6	6
7	57	57	8	8
8	73	73	9	9
9	91	91	10	10
11	133	133	12	12
13	183	183	14	14
16	273	273	17	17
17	307	307	18	18
19	381	381	20	20
23	553	553	24	24

#### 5. Conclusion

In conclusion, the study was able to establish a number of new properties for symmetric BIBD  $\lambda = 1$ . First, the study established that for the design to exist then the number of treatments  $v$  must be odd. Secondly for the design to exist, then the function  $4v - 3$  must result into a perfect odd square. Lastly, the total number of treatments  $v$  must be given by the function  $v = S^2 + S + 1$  where,  $S$  is a prime number or power of a prime number. In term terms of appropriate construction method for this class of design, the study established that the class of BIBD could be constructed using projective geometry  $PG(2, S)$ .

#### References

- [1] Akra, U. P., Akpan, S. S., Ugbe, T. A. and Ntekim, O. E. (2021). *Finite Euclidean Geometry Approach for Constructing Balanced Incomplete Block Design (BIBD)*. Asian Journal of Probability and Statistics. 11 (4): 47-59.
- [2] Alabi, M. A. (2018). *Construction of balanced incomplete block design of lattice series I and II*. International Journal of Innovative Scientific and Engineering Technologies Research. 2018; 6 (4): 10-22.
- [3] Alam, N. M. (2014). *On Some Methods of Construction of Block Designs*. I. A. S. R. I, Library Avenue, New Delhi-110012.
- [4] Bose, R. C. (1939), *On the construction of balanced incomplete block designs*. Annals of Eugenics, Vol. 9, pp. 353-399.
- [5] Bose, R. C., Shrikhande, S. S., and Parker, E. T. (1960). Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture. Canadian Journal of Mathematics, 12, 189-203.

- [6] Dey, A. (2010). *Incomplete Block Designs*. World Scientific Publishing Co. Pte. Ltd. Warren Street. U.S.A.
- [7] Greig, M., and Rees, D. H. (2003). Existence of balanced incomplete block designs for many sets of treatments. *Discrete Mathematics*, 261 (1-3), 299-324.
- [8] Goud T. S. and Bhatra, C. N. Ch. (2016). Construction of Balanced Incomplete Block Designs. *International Journal of Mathematics and Statistics Invention*. 4 (1) 2321-4767.
- [9] Hinkelmann, K. and Kempthorne, O. (2005). *Design and Analysis of Experiments*. John Wiley and Sons, Inc., Hoboken, New Jersey.
- [10] Hsiao-Lih, J., Tai-Chang, H. and Babul, M. H. (2007). *A study of methods for construction of balanced incomplete block design*. *Journal of Discrete Mathematical Sciences and Cryptography* Vol. 10 (2007), No. 2, pp. 227–243.
- [11] Janardan, M. (2018). *Construction of balanced incomplete block design: an application of galois field*. *Open Science Journal of Statistics and Application*. 2013; 5 (3): 32-39.
- [12] Kageyama, S. (1980). *On properties of efficiency balanced designs*. *Communication in Statistics*, A 9 (6), 597-616.
- [13] Khare, M. and W. T. Federer (1981). *A simple construction procedure for resolvable incomplete block designs for any number of treatments*. *Biom. J.*, 23, 121–132.
- [14] Mahanta, J. (2018). *Construction of balanced incomplete block design: An application of Galois field*. *Open Science Journal of Statistics and Application*.
- [15] Mandal, B. N. (2015). *Linear Integer Programming Approach to Construction of Balanced Incomplete Block Designs*. *Communications in Statistics - Simulation and Computation*, 44: 6, 1405-1411, DOI: 10.1080/03610918.2013.821482.
- [16] Montgomery, D. C. (2019). *Design and analysis of experiment*. John Wiley and Sons, New York.
- [17] Neil, J. S. (2010). *Construction of balanced incomplete block design*. *Journal of Statistics and Probability*. 12 (5); 231–343.
- [18] Wan, Z. X. (2009). *Design theory*. World Scientific Publishing Company.
- [19] Yasmin, F., Ahmed, R. and Akhtar, M. (2015). Construction of Balanced Incomplete Block Designs Using Cyclic Shifts. *Communications in Statistics—Simulation and Computation* 44: 525–532. DOI: 10.1080/03610918.2013.784984.
- [20] Yates, F. (1936). *A new method of arranging variety trials involving a large number of varieties*. *J. Agric. Sci.*, 26, 424-445.