Mathematical modelling and steady state performance analysis of a Markovian queue with heterogeneous servers and working vacation

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Abstract: In the present paper, mathematical modeling for analyzing a Markovian queueing system with two heterogeneous servers and working vacation has been demonstrated. Keeping in view queueing situations in real life problems, here we consider service policy that initially both the heterogeneous servers take vacation when there are no customers waiting for service in the queue; however, after this server 1 is always available but the other goes on vacation whenever server 2 is idle. The vacationing server however, returns to serve at a low rate as an arrival finds the other server busy. Busy period analysis for the working vacation model with heterogeneous servers has been worked out. Performance measures of the Markovian queueing system with varying parameters have been explored under steady state using matrix geometric method. Finally, based on sensitivity analysis of the performance measures, conclusive observations have been focused.

Keywords: Markovian queue, working vacation (WV), Bernoulli vacation, heterogeneous servers, algorithmic approach, matrix geometric solution, steady state performance measures

1. Introduction

We analyze an M/M/2 Markovian queue with working vacations (WVs), in which the server works with variable service rates rather than completely stops service during his vacation period. Such a vacation is called a working vacation. Each server starts a vacation when the system is empty at his service completion epoch. If server 1 returns from vacation and find server 2 is busy and there is no customers in the system then, he stays idle and ready for serving new arrivals. If the server 2 returns from a WV to find the non-empty system, he immediately switches to original service rate.

The queueing systems with server vacations or WVs have been investigated by many researchers. Past work may be divided into two categories: (i) the case of server vacation and (ii) the case of working vacation (WV). In the case of server vacation, the readers are referred to the survey paper by Doshi [4] and monograph of Takagi [26]. The works of Dosi [5], Maurya [16] and Takagi [26] focus on a single server with vacation model. As for multiple server system with vacations Zhang and Tian [28,29] recently contributed for plenty analysis of M/M/c with synchronous multiple/single vacations of partial servers. In the case of working vacation, Servi and Finn [25] first examined an M/M/1 queue with multiple WVs where inter-arrival times, service times during service period, and vacation times are all exponentially distributed. They developed the explicit formulae for the mean and variance number of customers in the system, and the mean and variance waiting time in the system. Later Wu and Takagi [26] extended Servi and Finn's [25] discuss the model M/M/1/WV queue to an M/G/1/WV queue. Kim et al. [8] have generalized the research work of the model of Servi & Finn [25] to the M/G/1 queue.

In queueing models vacations can be classified as single and multiple server involving single vacation and multiple vacations. The server may take a vacation at a random time, after serving at most K customers or after all the customers in the queue are served. The queueing systems with single or multiple vacations have been introduced by Levy and Yachiali [12]. The literature about vacation models is growing rapidly which includes survey papers by Teghem [23], Doshi [4,5] and the monograph by Takagi [26]. We can

find general models in Tian and Zhang [24]. In 2007, Li and Tian [8] first introduced vacation interruption policy and studied an M/M/1 queue. Recently, Ammar [1] has explored a new elegant explicit solution for a two heterogeneous servers queue with impatient behavior. In Chuen-Horng and Jau-chuan [3] considered an M/M/R queue with vacations, in which the server works with different service rates rather than completely terminates service during his vacation period. Li et al. [14] analyzed an M/M/1 retrial queue with WVs, vacation interruption, Bernoulli feedback, and N-policy simultaneously. Furthermore, research works of Maurya [17,18,19,20] focus on bulk arrival single server retrial queueing model with second phase optional service and Bernoulli vacation. During the WV period, customers can be served at a lower rate. Using the matrix-analytic method, we get the necessary and sufficient condition for the system to be stable.

The literature shows that the study on multi-server queueing systems generally assumes the servers to be homogeneous in which the individual service rates are the same for all the servers in the system. This assumption may be valid only when the service process is mechanically or electronically controlled. In a queueing system with human servers, the above assumption can hardly realized. However, recently Maurya [15,18] considered an interdependent M/M/1:(∞;GD) queueing model with bivariate Poisson process and controllable arrival rates. It is common to observe server rendering service to identical jobs at different service rates. This reality leads to modeling such multi-server queueing systems with heterogeneous servers that is the service time distributions may be different for different servers. Levy and Yechiali [11] have discussed the vacation policy in a multi-server Markovian queue and they considered a model with 's' homogeneous servers and exponentially distributed vacation times. Using partial generating function technique, the system size has been obtained. Kao and Narayanan [7] have discussed the M/M/s queue with multiple vacations of the servers using a matrix geometric approach. Gray et al. [6] have discussed a single counter queueing model involving multiple servers with multiple vacations. Tyagi et al. [9] have discussed an M/M/s queue with multiple vacation and 1-limited service. Neuts and Lucantoni [21] have analyzed the M/M/s queueing systems where the servers are subject to random breakdowns and repairs. Baba [2] extended work of Servi and Finn [25] with M/M/1/WV queue to a GI/M/1/WV queue. They assumed not only the general independent arrival but they also assumed service times during service period, service times during vacation period as well as vacation times following exponential distribution. Furthermore, Baba [2] derived the steady- state system length distributions at arrival and arbitrary epochs.

Neuts and Takahashi [22] observed that for queueing systems with more than two heterogeneous servers analytical results are intractable and only algorithmic approach could be used to study the steady state behavior of the system. Krishna Kumar and Pavai Madheswari [9] analyzed M/M/2 queueing system with heterogeneous servers where the servers go on vacation in the absence of customers waiting for service.Based on this observation, Krishnamoorthy and Sreenivasan [10] analyzed an M/M/2 queueing system with heterogeneous servers where one server remains idle but the other goes on vacation in the absence of waiting customers. In this paper we discuss an M/M/2 queueing system with heterogeneous servers where both servers go on vacation in the absence of customers, for the remaining times server 1 is always in the system and server 2 go for vacation whenever it is idle.

2. Quasi Birth and Death Process Model

Consider an M/M/2 queueing system with two heterogeneous servers. Arrivals of customers follow a Poisson process with parameter λ . let μ_1 and μ_2 be the service rates of server1 and server2 respectively, where $\mu_1 \neq \mu_2$. The duration of vacation periods are assumed to be independent and identically distributed exponential random variables with parameters θ_1 and θ_2 . During the vacation, if an arrival finds server 1 is busy then server 2 returns to serve the customer at a lower rate. To be precise server 2 serve this customers at the rate $\theta\mu_2, 0 < \theta \leq 1$. At this vacation gets over server 2 instantaneously switches over to the normal service rate μ_2 ,   upon completion of a service at low rate if no customer is waiting for service then go for vacation and if at least one customer is waiting for service then server 2 is busy with normal service rate. The arriving customers are served under the first-come-first-served (FCFS) discipline. The vacation queueing model with heterogeneous server under consideration can be formulated as a continuous time Markov chain (CTMC). The possible states of the system at any epoch are represented by (i, j) where $i \geq 0$ denotes the number of customers in the system and j=0,1,2,3 denotes the status of the servers. The state (0,0) represents there is no customers in the system and both servers are on vacation; after the state (0,0) server 1 is always available in the system the state $(i,1)$ represent ($i \geq 0$) customers are in the system and server 1 is busy in the system while server 2 is on vacation; the state $(i,2)$ represent $i \geq 0$ customers in the system and server 1 is busy and server 2 is in working vacation mode; the state $(i,3)$ represent $i \geq 0$ customers in the system and both servers are busy in the system with normal mode.

Let Q denotes the infinitesimal generator of the continuous time Markov chain (CTMC) corresponding to this Q and is in the format of a quasi-birth-and –death (QBD) process. Define the levels 0,1,2,…, as the set of the states 0={(0,0)}, $1 = \{(1,0),(1,1),(1,2)\}$, and $i = \{(i,0),(i,1),(i,2),(i,3)\}$ if $i \geq 2$. The state transition diagram of the system is as follows
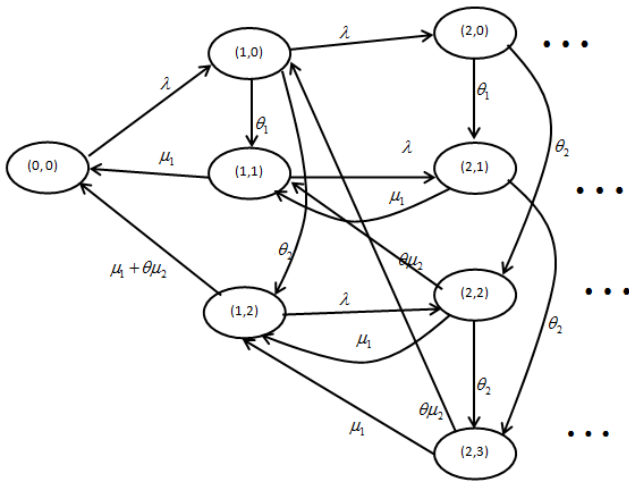
Figure 1. State transition diagram

Using the lexicographical sequence for the states the infinite generator can be written as

$$Q = \begin{bmatrix} B_{00} & B_{01} & & & & \\ B_{10} & B_{11} & B_{12} & & & \\ & B_{21} & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & A_2 & A_1 & A_0 \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

The sub-matrices A_0, A_1 and A_2 are of order 4x4 and are given by

$$A_0 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_1 + \theta\mu_2 & 0 \\ 0 & 0 & 0 & \mu_1 + \mu_2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda+\theta_1+\theta_2) & \theta_1 & 0 & \theta_2 \\ 0 & -(\lambda+\mu_1+\theta_2) & 0 & \theta_2 \\ 0 & 0 & -(\lambda+\mu_1+\theta\mu_2+\theta_2) & \theta_2 \\ 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2) \end{bmatrix}$$

The boundary matrices are defined by

$$B_{00} = -\lambda, \; B_{01} = (\lambda, 0, 0), \; B_{10} = [0, \mu_1, (\mu_1+\theta\mu_2)]^T$$

$$B_{11} = \begin{bmatrix} -(\lambda+\theta_1+\theta_2) & \theta_1 & \theta_2 \\ 0 & -(\lambda+\mu_1) & 0 \\ 0 & 0 & -(\lambda+\mu_1+\theta\mu_2) \end{bmatrix}$$

$$B_{12} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & \mu_1 & \theta\mu_2 \\ 0 & \mu_2 & \mu_1 \end{bmatrix}$$

Define the matrix $A = A_0 + A_1 + A_2$. Then A is 4x4 matrix and it can be written as

$$A = \begin{bmatrix} -(\theta_1+\theta_2) & \theta_1 & \theta_2 & 0 \\ 0 & -\theta_2 & 0 & \theta_2 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.1. Rate Matrix

To analyze this QBD process, a very important matrix in evaluating the performance measures is the matrix R. it is known as the rate matrix of the Markov chain Q and it has the minimal non-negative solution of the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0 \qquad (2.1)$$

Since the matrices A_0, A_1, and A_2 are of order 4x4 upper triangular, R is also a 4x4 upper triangular matrix.

Lemma 2.1

(a) The quadratic equation

$\mu_1 z^2 - (\lambda+\mu_1+\theta_2)z + \lambda = 0$ includes the service rate μ_1 of server 1 and the vacation parameter θ_2 of server 2.

Therefore, the above quadratic equation has two different real roots $r_1 < r_1^*$

and $0 < r_1 < 1$, $r_1^* > 1$

where

$$r_1 = \frac{1}{2\mu_1}\left((\lambda+\mu_1+\theta_2) - \sqrt{(\lambda+\mu_1+\theta_2)^2 - 4\lambda\mu_1}\right)$$

and

$$r_1^* = \frac{1}{2\mu_1}\left((\lambda+\mu_1+\theta_2) + \sqrt{(\lambda+\mu_1+\theta_2)^2 - 4\lambda\mu_1}\right)$$

(b) The quadratic equation

$(\mu_1+\theta\mu_2)z^2 - (\lambda+\mu_1+\theta\mu_2+\theta_2)z + \lambda = 0$ includes the service rate μ_1 of server 1 and the working vacation parameter $\theta\mu_2$ of server 2.

Therefore, the above quadratic equation has two different real roots $r_2 < r_2^*$

and $0 < r_2 < 1$, $r_2^* > 1$

where,

$$r_2 = \frac{1}{2(\mu_1+\theta\mu_2)}\left((\lambda+\mu_1+\theta\mu_2+\theta_2) - \sqrt{(\lambda+\mu_1+\theta\mu_2+\theta_2)^2 - 4\lambda(\mu_1+\theta\mu_2)}\right)$$

and

$$r_2^* = \frac{1}{2(\mu_1+\theta\mu_2)}\left((\lambda+\mu_1+\theta\mu_2+\theta_2) + \sqrt{(\lambda+\mu_1+\theta\mu_2+\theta_2)^2 - 4\lambda(\mu_1+\theta\mu_2)}\right)$$

(c) The quadratic equation

$(\mu_1+\mu_2)z^2 - (\lambda+\mu_1+\mu_2)z + \lambda = 0$ includes the two service rates μ_1 and μ_2 of server 1 and server 2 respectively, but it

does not include the parameters θ_1 and θ_2. From this it is clear that the above quadratic is for the case when both two servers are busy in service and it has two different real roots $r_3 = \rho = \lambda(\mu_1 + \mu_2)^{-1}$ and $r_3^* = 1$.

From the matrix R, we find that the spectral radius $sp(R) = \max(r_{0,0}, r_{1,1}, r_{2,2}, r_{3,3}) < 1$ if and only if $\rho < 1$. Hence, we can prove that $\rho < 1$ is the necessary and sufficient condition for that the stability of the process $\{L(t), J(t), t \geq 0\}$ is to be positive recurrent (see Neuts and Lucantoni [21]).

Theorem 2.1

If $\rho < 1$, the matrix equation (2.1) has the minimal non-negative solution as follows:

$$R = \begin{bmatrix} r_0 & r_{0,1} & 0 & r_{0,3} \\ 0 & r_1 & 0 & r_{1,3} \\ 0 & 0 & r_2 & r_{2,3} \\ 0 & 0 & 0 & \rho \end{bmatrix}$$

Where symbols used in the matrix are defined as following:

$$R^2 A_2 = \begin{bmatrix} 0 & \mu_1 \sum_{i=0}^{1} a_{0,i} a_{i,1} & (\mu_1 + \theta\mu_2) \sum_{i=0}^{2} a_{0,i} a_{i,2} & (\mu_1 + \mu_2) \sum_{i=0}^{3} a_{0,i} a_{i,3} \\ 0 & \mu_1 a_{1,1}^2 & (\mu_1 + \theta\mu_2) \sum_{i=1}^{2} a_{1,i} a_{i,2} & (\mu_1 + \mu_2) \sum_{i=1}^{3} a_{1,i} a_{i,3} \\ 0 & 0 & (\mu_1 + \theta\mu_2) a_{2,2}^2 & (\mu_1 + \mu_2) \sum_{i=2}^{3} a_{2,i} a_{i,3} \\ 0 & 0 & 0 & (\mu_1 + \mu_2) a_{3,3}^2 \end{bmatrix}$$ (2.3)

$$RA_1 = \begin{bmatrix} -(\lambda+\theta_1+\theta_2)a_{0,0} & a_{0,0}\theta_1+-(\lambda+\mu_1+\theta_2)a_{0,1} & -(\lambda+\mu_1+\theta\mu_2+\theta_2)a_{0,2} & \theta_2\sum_{i=0}^{2}a_{0,i}-(\lambda+\mu_2+\mu_1)a_{0,3} \\ 0 & -(\lambda+\mu_1+\theta_2)a_{1,1} & -(\lambda+\mu_1+\theta\mu_2+\theta_2)a_{1,2} & \theta_2\sum_{i=1}^{2}a_{1,i}-(\lambda+\mu_2+\mu_1)a_{1,3} \\ 0 & 0 & -(\lambda+\mu_1+\theta\mu_2+\theta_2)a_{2,2} & \theta_2 a_{2,2}-(\lambda+\mu_2+\mu_1)a_{2,3} \\ 0 & 0 & 0 & -(\lambda+\mu_2+\mu_1)a_{3,3} \end{bmatrix}$$ (2.4)

Substituting (3),(4) and A_0 into (1) gives the following set of equations

$$-(\lambda + \theta_1 + \theta_2)a_{0,0} + \lambda = 0$$ (2.5)

$$\mu_1 a_{1,1}^2 - (\lambda + \mu_1 + \theta_2)a_{1,1} + \lambda = 0$$ (2.6)

$$(\mu_1 + \theta\mu_2)a_{2,2}^2 - (\lambda + \theta\mu_2 + \mu_1 + \theta_2)a_{2,2} + \lambda = 0$$ (2.7)

$$(\mu_1 + \mu_2)a_{3,3}^2 - (\lambda + \mu_1 + \mu_2)a_{3,3} + \lambda = 0$$ (2.8)

$$\mu_1 \sum_{i=0}^{1} a_{0,i} a_{i,1} + a_{0,0}\theta_1 - (\lambda + \mu_1 + \theta_2)a_{0,1} = 0$$ (2.9)

$$r_{0,0} = \frac{\lambda}{\lambda + \theta_1 + \theta_2}, \quad r_{0,1} = \frac{\theta_1 r_{0,0}}{\mu_1(r_{1,1}^* - r_{0,0})}, \quad r_{0,2} = \frac{\theta_2 r_{0,0}}{\theta\mu_2(r_{2,2}^* - r_{0,0})},$$

$$r_{0,3} = \frac{\left[\left(1 + \dfrac{a_{1,1}}{(1-a_{1,1})}\right)\dfrac{\theta_1\theta_2 a_{0,0}}{\mu_1(a_{1,1}^* - a_{0,0})}\right] + \theta_2 a_{0,0}}{(\mu_1 + \mu_2)(1 - a_{0,0})},$$

$$r_{1,3} = \frac{r_{1,1}\theta_2}{(\mu_1 + \mu_2)(1 - r_{1,1})},$$

$$r_{2,3} = \frac{\theta_2 a_{2,2}}{(\mu_1 + \mu_2)(1 - a_{2,2})}.$$

Proof:

Since the coefficient matrices of equation (1) are all upper triangular, so let

$$R = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & 0 & a_{3,3} \end{bmatrix}$$ (2.2)

$$(\mu_1 + \theta\mu_2)\sum_{i=0}^{2} a_{0,i} a_{i,2} + a_{0,0}\theta_2 - (\lambda + \theta\mu_2 + \mu_1 + \theta_2)a_{0,2} = 0$$ (2.10)

$$(\mu_1 + \theta\mu_2)\sum_{i=1}^{2} a_{1,i} a_{i,2} - (\lambda + \theta\mu_2 + \mu_1 + \theta_2)a_{1,2} = 0$$ (2.11)

$$(\mu_1 + \mu_2)\sum_{i=0}^{3} a_{0,i} a_{i,3} + \theta_2 \sum_{i=0}^{2} a_{0,i} - (\lambda + \mu_1 + \mu_2)a_{0,3} = 0$$ (2.12)

$$(\mu_1 + \mu_2)\sum_{i=1}^{3} a_{1,i} a_{i,3} + \theta_2 \sum_{i=1}^{2} a_{1,i} - (\lambda + \mu_1 + \mu_2)a_{1,3} = 0$$ (2.13)

$$(\mu_1 + \mu_2)\sum_{i=2}^{3} a_{2,i} a_{i,3} + \theta_2 a_{2,2} - (\lambda + \mu_1 + \mu_2)a_{2,3} = 0$$ (2.14)

From equation (2.5), we get

$$a_{0,0} = \frac{\lambda}{\lambda + \theta_1 + \theta_2}, \qquad (2.15)$$

Solving equation (2.9), we get

$$a_{0,1} = \frac{\theta_1 a_{0,0}}{\mu_1(a_{1,1}^* - a_{0,0})}, \qquad (2.16)$$

Solving equation (2.11), we get

$$a_{1,2} = 0 \qquad (2.17)$$

Using (2.17) in equation (2.10), we get

$$a_{0,2} = 0 \qquad (2.18)$$

Using equation (2.16) in equation (2.12), we get

$$a_{0,3} = \frac{\left[\left(1 + \frac{a_{1,1}}{(1 - a_{1,1})}\right)\frac{\theta_1\theta_2 a_{0,0}}{\mu_1(a_{1,1}^* - a_{0,0})}\right] + \theta_2 a_{0,0}}{(\mu_1 + \mu_2)(1 - a_{0,0})}, \qquad (2.19)$$

Making use of equation (2.17) in equation (2.13), we get

$$a_{1,3} = \frac{\theta_2 a_{1,1}}{(\mu_1 + \mu_2)(1 - a_{1,1})}, $$

Finally, from equation (2.14), we have

$$a_{2,3} = \frac{\theta_2 a_{2,2}}{(\mu_1 + \mu_2)(1 - a_{2,2})}, $$

It is clear that the above equations have unique non-negative solution. Therefore, this non-negative solution must be the minimal.

3. Stationary Distribution

Let L and J be the stationary random variables for the queue length and the status of the servers. Denote the stationary probability by

$$x_{ij} = P\{L = i, J = j\}$$

$$= \lim_{t \to \infty} P\{L(t) = i, J(t) = j\}, \ (i, j) \in \Omega.$$

Under the stability condition $\rho < 1$, the stationary probability vector x of the generator Q exists. This stationary probability vector x is partitioned as $x = (x_0, x_1, x_2, ...)$ where x_0 is a scalar.

$$x_1 = (x_{10}, x_{11}, x_{12}) \text{ and } x_i = (x_{i0}, x_{i1}, x_{i2}, x_{i3}) \text{ for } i \geq 2$$

Based on the matrix geometric solution method in (see Neuts and Lucantoni [21]), the stationary probability vector x is given by

$$x_0 B_{00} + x_1 B_{10} = 0, \qquad (3.1)$$

$$x_0 B_{01} + x_1 B_{11} + x_2 B_{21} = 0, \qquad (3.2)$$

$$x_1 B_{12} + x_2 (A_1 + RA_2) = 0, \qquad (3.3)$$

$$x_i = x_2 R^{i-2}, \ i = 3, 4, 5, ... \qquad (3.4)$$

and the normalizing equation

$$x_0 + x_1 e_1 + x_2 (I - R)^{-1} e_2 = 1 \qquad (3.5)$$

Where I is a 4x4 identity matrix, e_1 is a 3x1 column vector and e_2 is a 4x1 column vector with all their elements equal to one.

$$A_1 + RA_2 = \begin{bmatrix} -(\lambda + \theta_1 + \theta_2) & \left(1 + \frac{r_0}{r_1^* - r_0}\right)\theta_1 & \theta_2 & \left(\frac{1}{1 - r_0}\right)\left[\left(1 + \frac{r_1}{1 - r_1}\right)\left(\frac{\theta_1\theta_2 r_0}{\mu_1(r_1^* - r_0)}\right) + \theta_2 r_0\right] \\ 0 & -h_1 & \theta_2 & \left(\frac{r_1\theta_2}{1 - r_1}\right) \\ 0 & 0 & -h_2 & \left(1 + \frac{r_2}{1 - r_2}\right)\theta_2 \\ 0 & 0 & 0 & -(\mu_1 + \mu_2) \end{bmatrix}$$

Now, it is fairly easy that the equations (3.1),(3.2) and (3.3) can be written as the set of equations;

$$-\lambda x_0 + \mu_1 x_{11} + (\mu_1 + \theta\mu_2)x_{12} = 0 \qquad (3.6)$$

$$\lambda x_0 - (\lambda + \theta_1 + \theta_2)x_{10} = 0 \qquad (3.7)$$

$$\theta_1 x_{10} - (\lambda + \mu_1)x_{11} + \mu_1 x_{21} + \mu_1 x_{22} + \mu_2 x_{23} = 0 \quad (3.8)$$

$$\theta_2 x_{10} - (\lambda + \theta\mu_2 + \mu_1)x_{12} + \theta\mu_2 x_{22} + \mu_1 x_{23} = 0 \quad (3.9)$$

$$\lambda x_{10} - (\lambda + \theta_1 + \theta_2)x_{20} = 0 \qquad (3.10)$$

$$\lambda x_{11} + \left(1 + \frac{r_0}{(r_1^* - r_0)}\right)x_{20} - h_1 x_{21} = 0 \qquad (3.11)$$

$$\lambda x_{12} + \theta_2 x_{20} + \theta_2 x_{21} - h_2 x_{22} = 0 \qquad (3.12)$$

$$\phi x_{20} + \left(\frac{r_1\theta_2}{(1 - r_1)}\right)x_{21} + \left(1 + \frac{r_2}{1 - r_2}\right)\theta_2 x_{22} - (\mu_1 + \mu_2)x_{23} = 0 \quad (3.13)$$

Where,

$$\phi = \left(\frac{1}{1 - r_0}\right)\left[\left(1 + \frac{r_1}{1 - r_1}\right)\left(\frac{\theta_1\theta_2 r_0}{\mu_1(r_1^* - r_0)}\right) + \theta_2 r_0\right]$$

Solution of equation (3.7) can be easily found as following:

$$x_{10} = r_0 x_0$$

Solving equation (3.10), we get

$$x_{20} = r_0^2 x_0$$

Solving the remaining equations, we have

$$x_{11} = \left[\frac{\lambda - (\mu_1 + \theta\mu_2)\beta_1}{\mu_1} \right] x_0,$$

$$x_{12} = \beta_1 x_0$$

$$x_{21} = \beta_2 x_0$$

$$x_{22} = \beta_3 x_0$$

$$x_{23} = \beta_4 x_0$$

Where,

$$\beta_1 = \left[\frac{\phi_4\phi_6 + \phi_5 h_1 \mu_1}{\lambda\phi_6(\mu_1 + \theta\mu_2) - (\lambda\phi_3 + \phi_2 h_2)(\mu_1 h_1)} \right]$$

$$\beta_2 = \left[\frac{\phi_4 - \lambda\beta_1(\mu_1 + \theta\mu_2)}{(\mu_1 h_1)} \right]$$

$$\beta_3 = \left[\frac{\lambda\beta_1 + \theta_2 r_0^2 + \theta_2\beta_2}{h_2} \right]$$

$$\beta_4 = \left[\frac{\phi r_0^2 + \left(\frac{r_1\theta_2}{1-r_1} \right)\beta_2 + \left(1 + \frac{r_2\theta_2}{1-r_2} \right)\beta_3}{(\mu_1 + \mu_2)} \right]$$

$$\phi_1 = \left[(\mu_1 + \mu_2)\theta_2 r_0 + \mu_1\phi r_0^2 \right]$$

$$\phi_2 = -\left[(\mu_1 + \mu_2)(\lambda + \mu_1 + \theta\mu_2) \right]$$

$$\phi_3 = \left[(\mu_1 + \mu_2)\theta\mu_2 + \left(1 + \frac{r_2}{1-r_2} \right)\theta_2\mu_1 \right]$$

$$\phi_4 = \left[\lambda^2 + \left(1 + \frac{r_0}{r_1^* - r_0} \right)\theta_1\mu_1 r_0^2 \right]$$

$$\phi_5 = \left[\phi_1 h_2 + \theta_2\phi_3 r_0^2 \right]$$

$$\phi_6 = \left[\theta_2\phi_3 + \left(\frac{r_1\theta_2\mu_1 h_2}{1-r_1} \right) \right]$$

$$h_1 = \frac{1}{2}\left[(\lambda + \mu_1 + \theta_2) + \sqrt{(\lambda + \mu_1 + \theta_2)^2 - 4\lambda\mu_1} \right]$$

$$h_2 = \frac{1}{2}\left[(\lambda + \mu_1 + \theta\mu_2) + \sqrt{(\lambda + \mu_1 + \theta\mu_2 + \theta_2)^2 - 4\lambda(\mu_1 + \theta\mu_2)} \right]$$

$$(I-R)^{-1} = \begin{bmatrix} \left(\frac{1}{1-r_0}\right) & \left(\frac{r_{0,1}}{(1-r_0)(1-r_1)}\right) & 0 & \left(\frac{[r_{1,3}r_{0,1} + r_{0,3}(1-r_1)]}{(1-r_0)(1-r_1)(1-\rho)}\right) \\ 0 & \left(\frac{1}{1-r_1}\right) & 0 & \left(\frac{r_{1,3}}{(1-r_1)(1-\rho)}\right) \\ 0 & 0 & \left(\frac{1}{1-r_2}\right) & \left(\frac{r_{2,3}}{(1-r_2)(1-\rho)}\right) \\ 0 & 0 & 0 & \left(\frac{1}{1-\rho}\right) \end{bmatrix} \quad (3.14)$$

Substituting (3.14) and I where, I be the unit matrix of order 4x4 in (3.5), we get

$$y = \begin{bmatrix} 1 + r_0 + \left\{ \frac{\lambda - (\mu_1 + \theta\mu_2)\beta_1}{\mu_1} \right\} + \beta_1 + \left(\frac{1}{(1-r_{3,3})} \right)\beta_4 \\ + \left\{ \frac{1}{(1-r_{0,0})} + \frac{r_{0,1}}{(1-r_{0,0})(1-r_{1,1})} + \frac{r_{1,3}r_{0,1} + r_{0,3}(1-r_{1,1})}{(1-r_{0,0})(1-r_{1,1})(1-r_{3,3})} \right\} r_0^2 \\ + \left\{ \frac{1}{(1-r_{1,1})} + \frac{r_{1,3}}{(1-r_{1,1})(1-r_{3,3})} \right\}\beta_2 + \left\{ \frac{1}{(1-r_{2,2})} + \frac{r_{2,3}}{(1-r_{2,2})(1-r_{3,3})} \right\}\beta_3 \end{bmatrix}$$

$$x_0 = y^{-1}$$

Let L denote the stationary queue length at an arbitrary epoch. Therefore, the mean, second moment and variance of the number of customers in the system can be obtained as

$$E(L) = x_1 e_1 + 2x_2 e_2 + x_2 R\left[(I-R)^{-2} + 2(I-R)^{-1} \right]e_2$$

$$E(L^2) = x_1 e_1 + 4x_2 e_2 + x_2\left[(I-R)^{-1} + (I-R)^{-2} + 2(I-R)^{-3} - 4I \right]e_2$$

and $\text{var}(L) = E(L^2) - [E(L)]^2$

4. Busy Period Analysis of the Model

For the working vacation model with heterogeneous servers, the busy period is defined to be the interval between the arrival of a customer to an empty system and first epoch thereafter when the system becomes empty again. Thus, the busy period is the first passage time from state (1,0) to state (0,0). For the working vacation model, busy cycle for the system is the time interval between two successive departures, which leave the system empty. Therefore, the busy cycle is the first return time to state (0,0) with at least one visit to any other state.

To discuss busy period analysis, the notion of fundamental period (14) should be briefly reviewed. For the QBD process described above, starting from a state in level I, where $i \geq 3$, the first passage time to a state in level i-1 constitutes a fundamental period. The cases i=2, i=1, and i=0 corresponding to the boundary states need to be discussed separately. Because of the structure of the QBD process, the first passage time distribution is invariant in i.

Let $G_{jj'}(k,t)$ denote the conditional probability that a QBD process starting in the state (I,j)at time t=0, the first visit to level i-1 occurs no later than time t, into the state (i-1,j') and exactly k transitions occur to the left during the first passage time.

The matrix representation of joint transforms of $G(z,s)$

is defined by

$$G(z,s) = \sum_{k=1}^{\infty} z^k \int_0^{\infty} e^{-st} dG(k,t) \text{ for } |Z| \leq 1, \text{ Re } s \geq 0 \quad (4.1)$$

$G(z,s)$ satisfies the relation $G^n(z,s) = \left[G^1(z,s) \right]^n$ for $n \geq 1$ (4.2)

and define $G^1(z,s) = G(z,s)$

then the matrix $G(z,s)$ satisfies the equation

$$G(z,s) = z(sI - A_1)^{-1} A_2 + (sI - A_1)^{-1} A_0 G^2(z,s) \quad (4.3)$$

To discuss the boundary conditions for the vacation model, for I = 1,2, let $G_{jj'}^{(i,i-1)}(k,t)$ denote the conditional probability that a QBD process, starting in the state (I, j) at time 0, reaches the level i-1 for the first time nolater than t, after exactly k transitions to the left and does so by entering the state (i-1, j'). let $G^{(i,i-1)}(z,s)$ denote the transform matrix corresponding to $G^{(i,i-1)}(k,t)$. Using the similar argument in (4.3), we get

$$G^{(2,1)}(z,s) = z(sI - A_1)^{-1} B_{21} + (sI - A_1)^{-1} A_0 G(z,s) G^{(2,1)}(z,s), \quad (4.4)$$

$$G^{(1,0)}(z,s) = z(sI - B_{11})^{-1} B_{10} + (sI - B_{11})^{-1} B_{12} G^{(2,1)}(z,s) G^{(1,0)}(z,s), \quad (4.5)$$

$$G^{(0,0)}(z,s) = \left[\frac{\lambda}{s+\lambda}, 0, 0 \right] G^{(1,0)}(z,s), \quad (4.6)$$

Where $G^{(0,0)}(z,s)$ is the joint transform of the recurrence time to state (0,0) with at least one visit to a state other than state (0,0). Note that $G^{(1,0)}(z,s)$ is of 3x1. The Laplace Stieltjes transform (LST) for the length of a busy period is then given by the first element of $G^{(1,0)}(1+,s)$, that is, the element corresponds to the first passage from state (1,0) to the state (0,0). The busy cycle comprises an idle period and a busy period. The LST for the length of a busy cycle is given by $G^{(0,0)}(1+,s)$.

Let the matrices

$$G = \lim_{\substack{z \to 1- \\ s \to 0+}} G(z,s), \ G_{2,1} = \lim_{\substack{z \to 1- \\ s \to 0+}} G^{(2,1)}(z,s), \ G_{1,0} = \lim_{\substack{z \to 1- \\ s \to 0+}} G^{(1,0)}(z,s), \text{ and}$$

$$G_{0,0} = \lim_{\substack{z \to 1- \\ s \to 0+}} G^{(0,0)}(z,s). \quad (4.7)$$

The positive recurrence of the process Q implies that G, $G_{2,1}$, $G_{1,0}$ and $G_{0,0}$ are all stochastic and $G_{0,0} = (1,1,1)^T$. Let $C_0 = (-A_1)^{-1} A_2$, $C_2 = (-A_1)^{-1} A_0$. in (14) it has been proved that G is the minimal non negative solution to the equation $G = C_0 + C_2 G^2$.

Taking the $\lim_{\substack{z \to 1- \\ s \to 0+}}$ on both sides of the equations (4.4), (4.5), (4.6) and using (4.7), we get

$$G_{2,1} = -(A_1 + A_0 G)^{-1} B_{21}, \quad (4.8)$$

$$G_{1,0} = -(B_{11} + B_{12} G_{21})^{-1} B_{10}, \quad (4.9)$$

$$G_{0,0} = [1\ 0\ 0] G_{10}. \quad (4.10)$$

If we found the value of the matrix R then matrix G can be computed using the following result (15),

$$G = -(A_1 + RA_2)^{-1} A_2. \quad (4.11)$$

Let $M_1 = \left. \frac{\partial G(z,s)}{\partial s} \right|_{z=1, s=0}$ it has been proved (14) that the matrix M_1 can be computed by successive substitutions in

$$M_1 = -A_1^{-1} G + C_2 (GM_1 + M_1 G),$$

With 0 as the starting value for M_1. For the boundary states, differentiating (4.4), (4.5) and (4.6) with respect to s and setting s=0, z=1, we get

$$M_{2,1} = -(A_1 + A_0 G)^{-1} (I + A_0 M) G_{2,1},$$

$$M_{1,0} = -(B_{11} + B_{12} G_{21})^{-1} (I + B_{12} M_{21}) G_{1,0},$$

It is clear that $M_{1,0}$ can be computed recursively starting with M_1 in decreasing order. Assuming $M_{0,0}$ as the mean recurrence time for state (0,0), a similar operation leads to

$$M_{0,0} = \left[\frac{1}{\lambda}\ 0\ 0 \right] G_{1,0} + [1\ 0\ 0] M_{1,0}.$$

Where

$$M_{2,1} = \left. \frac{\partial G^{(2,1)}(z,s)}{\partial s} \right|_{z=1, s=0}$$

$$M_{1,0} = \left. \frac{\partial G^{(1,0)}(z,s)}{\partial s} \right|_{z=1, s=0}$$

$$M_{0,0} = \left. \frac{\partial G^{(0,0)}(z,s)}{\partial s} \right|_{z=1, s=0}$$

Table 1. *Performance measures of the Markovian queueing system when* $\theta = 0.7, \theta_1 = 3, \theta_2 = 1$

	$\theta = 0.7, \theta_1 = 3, \theta_2 = 1$					
λ	$\mu_1 = 10, \mu_2 = 5$			$\mu_1 = 8, \mu_2 = 8$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	**Var(L)**
2	0.8055	1.7748	1.1260	0.8572	1.9081	1.1733
4	1.7254	5.7183	2.7414	1.8597	6.3006	2.8422
6	2.8105	12.7750	4.8762	3.0508	14.5064	5.1993
8	4.1633	25.3177	7.9847	4.4998	29.4524	9.2040
10	6.0124	49.9098	13.7611	6.3654	57.1470	16.6291
12	9.3096	115.8571	29.1889	9.1843	114.6471	30.2961

Table 2. *Performance measures of the Markovian queueing system with variable parameters*

λ	$\theta = 0.7, \theta_1 = 3, \theta_2 = 1$			$\theta = 0.7, \theta_1 = 1, \theta_2 = 3$		
	$\mu_1 = 5, \mu_2 = 10$			$\mu_1 = 10, \mu_2 = 5$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	Var(L)
2	1.0451	2.4825	1.3901	0.8797	1.9192	1.1452
4	2.3418	9.0528	3.5686	1.6970	5.2289	2.3490
6	3.8745	22.8721	7.8607	2.4972	10.1618	3.9259
8	5.7453	49.2668	16.2579	3.4201	17.9936	6.2963
10	8.2651	98.1197	29.8078	4.7140	33.0041	10.7822
12	12.5174	206.9532	50.2688	7.2500	77.5638	25.0016

It is remarked here that the mean length of a busy cycle is $M_{0,0}$. The first element of the column vector $M_{1,0}$ yields the mean length of a busy period, E(B). The mean length of a server vacation period, E(v) is $\dfrac{1}{\theta_1 + \theta_2}$. The mean length of a server busy period is obtained as E(B)-E(V).

Table 3. *Performance measures when* $\theta = 0.7, \theta_1 = 1, \theta_2 = 3$ *and* $\theta = 0.7, \theta_1 = 1, \theta_2 = 3$

λ	$\theta = 0.7, \theta_1 = 1, \theta_2 = 3$			$\theta = 0.7, \theta_1 = 1, \theta_2 = 3$		
	$\mu_1 = 8, \mu_2 = 8$			$\mu_1 = 5, \mu_2 = 10$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	Var(L)
2	0.8914	1.9297	1.1350	0.9752	2.1334	1.1824
4	1.7044	5.1609	2.2559	1.8498	5.6507	2.2290
6	2.4718	9.7623	3.6524	2.6646	10.6711	3.5712
8	3.3112	16.6156	5.6518	3.5903	18.6125	5.7220
10	4.3985	28.3871	9.0399	4.9158	34.1461	9.9812
12	6.2137	55.7571	17.1471	7.5645	80.8240	23.6019

Table 4. *Performance measures of the system when* $\theta = 0.7, \theta_1 = 2, \theta_2 = 2$

λ	$\theta = 0.7, \theta_1 = 2, \theta_2 = 2$					
	$\mu_1 = 10, \mu_2 = 5$			$\mu_1 = 8, \mu_2 = 8$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	Var(L)
2	0.8393	1.8341	1.1296	0.8685	1.8947	1.1404
4	1.6930	5.3562	2.4901	1.7434	5.4858	2.4463
6	2.5817	10.9024	4.2373	2.6285	10.9963	4.0875
8	3.6175	19.8615	6.7753	3.6109	19.4918	6.4533
10	5.0411	36.8464	11.4337	4.8623	34.0356	10.3935
12	7.7514	85.6532	25.5685	6.8842	66.4325	19.0400

Table 5. *Performance measures of the system when* $\theta = 0.7, \theta_1 = 2, \theta_2 = 2$

λ	$\theta = 0.7, \theta_1 = 2, \theta_2 = 2$		
	$\mu_1 = 5, \mu_2 = 10$		
	$E[L]$	$E[L^2]$	Var(L)
2	0.9908	2.2200	1.2383
4	1.9797	6.4819	2.5628
6	2.9588	13.1594	4.4048
8	4.0781	24.0372	7.4065
10	5.6392	44.5591	12.7591
12	8.6188	101.0972	26.8131

5. Conclusions

Here, steady state performance analysis of an M/M/2 Markovian queue with working vacations (WVs) has been explored successfully using matrix geometric method. The present study of the M/M/2 Markovian queue with working vacations (WVs) is an extended model studied by Maurya [16]. For different varying parameters, the mean queue length and its variance have been obtained in different tables 1-5; from which one can easily observe the varying trend of the mean queue length and its variance for the M/M/2 Markovian queue with working vacations (WVs) model. As far as its future scope is concerned, we suggest that the model can be further extended for more than two heterogeneous servers as well as the arrival pattern may take place in batches. Apart from this, the present study can also be extended with other concepts such as non-Markovian assumptions etc.

References

[1] Ammar Sherif I., Transient analysis of a two-heterogeneous servers queue with impatient Behavior, *Journal of the Egyptian Mathematical Society*, Vol. 22, pp. 90-95, 2014.

[2] Baba Y., Analysis of a GI/M/I queue with multiple working vacations, *Operations Research Letters*, Vol.33 (2), pp. 209-209, 2005.

[3] Chuen-Horng Lin and Jau-chuan Ke, Multi-server system with single working vacation, *Applied Mathematical Modelling*, Vol.33, pp. 2967-2977, 2009.

[4] Doshi B.T., Queueing systems with vacations – a survey, *Queueing Syst.*, Vol. 1, pp. 29–66, 1986.

[5] Doshi B.T., Single-server queues with vacations, Stochastic Analysis of Computer and   Communications Systems, (Edited by H. Takagi), pp. 217-265, *Elsevier*, Amsterdam, 1990.

[6] Gray W.J., Wang P.P. and Scott M., Single counter queueing models involving multiple servers, *Opsearch*, Vol. 34 (4), pp.283-296, 1997.

[7] Kao E.P.C. and Narayanan K.S., Analyses of an M/M/N queue with server's vacations, *European Journal of Operational Research*, Vol. 54, pp.256-266, 1991.

[8] Kim J.D., Choi D.W. and Chae K.C., Analysis of queue-length distribution of the M/G/1 with working Vacations (M/G/1/WV), Proceeding of the International Conference on Statistics and related, Honolulu, Hawaii, USA, 2003.

[9] Krishna Kumar B. and Pavai Madheswari S., An M/M/2 queueing system with Heterogeneous Servers and multiple vacation, *Mathematical and Computer Modeling*, Vol. 41, pp. 1415-1429, 2005.

[10] Krishnamoorthy A. and Sreenivasa C., An M/M/2 queueing system with heterogeneous servers including one with working vacation, *International Journal of Stochastic Analysis*, Hindawi Publishing Corporation, 2012

[11] Levy Y. and Yechiali U., An M/M/S queue with servers' vacations, *Infor*. Vol. 14, pp. 153-163, 1976.

[12] Levy Y. and Yechiali U., Utilization of idle time in an M/G/1 queueing system, *Management Science*, Vol. 22, pp. 202-211, 1975.

[13] Li J. and Tian N., The M/M/1 queue with working vacations and vacation interruptions, *Journal of Systems Science and Systems Engineering*, Vol.16 (1), pp.121-127, 2007.

[14] Li Tao, Liyuan Zhang and Shan Gao, M/M/1 retrial queue with working vacation interruptions and feedback under N-Policy, *Journal of Applied Mathematics*, Hindawi Publishing Corporation, 2014.

[15] Maurya V.N., Determination of expected busy periods in faster and slower arrival rates of an interdependent M/M/1:(∞; GD) queueing model with controllable arrival rates, *International Journal of Engineering Research and Technology*, Engineering Science & Research Support Academy (ESRSA) Publication, Vadodara, India, Vol.1(5), pp. 1-5, 2012.

[16] Maurya V.N., Mathematical modeling and performance analysis of single server two-state batch arrivals and batch service Markovian queue with multiple vacations, *American Journal of Modeling and Optimization*, Science & Education Publishing, USA, Vol. 2 (2), pp. 39-46, 2014.

[17] Maurya V.N., Maximum entropy analysis of $M^X/(G_1,G_2)/1$ retrial queueing model with second phase optional service and Bernoulli vacation schedule, *American Journal of Operational Research*, Scientific and Academic Publishing, Rosemead, California, USA, Vol. 3(1), pp. 1-12, 2013.

[18] Maurya V.N., On busy period of an interdependent M/M/1:(∞;GD) queueing model with bivariate Poisson process and controllable arrival rates, *IEEE Transactions*, pp. 243-246, 2010.

[19] Maurya V.N., Performance analysis of $M^X/(G_1,G_2)/1$ queueing model with second phase optional service and Bernoulli vacation schedule, *American Journal of Modeling and Optimization, Science and Education* Publishing, New York, USA, Vol. 2(1), pp. 01-07, 2014

[20] Maurya V.N., Sensitivity analysis on significant performance measures of bulk arrival retrial queueing $M^X/(G_1,G_2)/1$ model with second phase optional service and Bernoulli vacation

[schedule, *International Open Journal of Operations Research,* Academic and Scientific Publishing, New York, USA, Vol. 1(1), pp. 1-15, 2013.

[21] Neuts M.F. and Lucantoni D.M., A Markovian queue with N servers subject to breakdowns and repairs, *Management Science*, Vol. 25, pp. 849-861, 1979.

[22] Neuts M.F. and Takahashi Y., Asymptotic behaviour of the stationary distribution in the CI/PH/c queue with heterogeneous servers, Z. *Wahrseheinlickeitsth*, Vol. 57, pp.441-452, 1981.

[23] Servi L.D. and Finn S.G., M/M/1 queues with working vacations (M/M/1/WV), *Perform. Eval.*, Vol. 50, pp. 41–52, 2002.

[24] Takagi H., Queueing Analysis – A Foundation of Performance Evaluation Vacation and Priority Systems, Vol. 1, North-Holland, New York, 1991.

[25] Teghem J., Control of the service process in a queueing system, *European Journal of Operational Research*, Vol. 23, pp.141-158, 1986.

[26] Tian N. and Zhang Z.G., Vacation Queueing Models, *International Series in Operations Research & Management Science*, Vol. 93, Springer, New York, USA, 2006.

[27] Wu D.A. and Takagi H., M/G/1 queue with multiple working vacations, *Perform. Eval.*, Vol. 63, pp. 654–681, 2006.

[28] Zhang Z.G. and Tian N. Analysis of queueing systems with synchronous single vacation for some servers, *Queueing Syst.*, Vol. 45, pp. 161–175, 2003.

[29] Zhang Z.G. and Tian N., Analysis on queueing systems with synchronous vacations of partial servers, *Perform. Eval.*, Vol. 52, pp. 269–282, 2003.]

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