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Synchronization and Impulsive Control of Some Parabolic Partial Differential Equations

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Abstract: Novel equi-attractivity in large generalized non-linear partial differential equations were performed for the impulsive control of spatiotemporal chaotic. Attractive solutions of these general partial differential equations were determined. A proof for existence of a certain kind of impulses for synchronization such that the small error dynamics that is equi-attractive in the large is established. A comparative study between these general non-linear partial differential equations and the existent reported numerical theoretical models was developed. Several boundary conditions were given to confirm the theoretical results of the general non-linear partial differential equations. Moreover, the equations were applied to Kuramoto–Sivashinsky PDE′s equation; Grey–Scott models, and Lyapunov exponents for stabilization of the large chaotic systems with elimination of the dynamic error.

Keywords: Synchronization, Impulsive Control, Prabolic Partial Differential Equations

1. Introduction

The ordinary differential equations (ODE′s) theory were applied in science and engineering researches [1, 2, 3], for mathematical modeling of many physical phenomena. The impulsive control on basis of these equations was successfully applied for stabilization of the systems with chaotic behavior using small control impulses even if the chaotic signals and noise are unpredictable. For example, autonomous systems of ODE′s Lorenz and Chua oscillator systems [4, 5, 6, 7], and non-autonomous systems such as Duffings oscillator [8, 9], and where practical stability of the system is achieved in a small region of phase space instead of controlling the approach of chaotic system to an equilibrium position. The impulsive synchronization of two identical chaotic systems by ODE′s [10, 11, 12, 13], involved autonomous drive system, and response system. Samples of the state variables (synchronization impulses) of drive system at discrete time intervals were used to: 1) drive the response system, 2) impulsively control error between the two systems, 3) minimizing the dynamic error, and 4) an upper bound on time intervals between impulses is obtained. This synchronization was generalized to vary impulse intervals [14, 15, 16], where less conservative conditions on Lyapunov function are obtained meaning that, it is required to be non-increasing along a subsequence of switching. The impulsive synchronization was applied in secure communications [17, 18], analysis of impulsive control, and synchronization of chaotic systems extending the theory of impulsive differential equations to PDE′s [19, 20, 21, 22], giving several differential inequalities, asymptotic stability, and first order partial differential-functional equations using Lyapunov energy functions, and the numerical analysis of first order PDE′s [23, 24, 25], The general application of impulsive control and impulsive synchronization on spatiotemporal chaotic systems generated by continuous extended systems including synchronization of spatiotemporal chaotic systems generated by coupled non-linear oscillators using ODE′s [26, 27, 28], and impulsive synchronization of spatiotemporal chaotic systems using PDE′s [29, 30, 31], using a finite number of local tiny perturbations selected by an adaptive technique [32, 33, 34], or using an extended time-delay auto synchronization algorithm [35, 36], or synchronizing using a finite number of coupling signals in terms of local spatial averages [37, 38, 39], frequency and phase synchronization of two non-identical PDE′s [40, 41, 42]. Using high dimensional PDE′s involving multiple stable and unstable modes, so synchronization process is more difficult compared to synchronizing using low dimensional ODE′s. Most of coupling schemes for spatiotemporal synchronization are very difficult to implement experimentally because coupling must be applied at all spatial points simultaneously or some variable of driven system must be reset to new values at specific points in space [43, 44, 45, 46], However, these problems can be solved in impulsive synchronization in which much smaller subset of points are driven impulsively. The complex behavior of spatiotemporal synchronization, and long time consumed to solve PDE′s numerically slow down synchronization process generate problems in implementation. This character of PDE′s represent advantages in masking information for secure communication (e.g., many more frequencies are involved in mask on using PDE) and security of information transmission increased [47, 48, 49], and multichannel spread-spectrum communication become efficient since a large number of informative signals can be transmitted

and received simultaneously. The implementation of impulsive synchronization of spatiotemporal chaos in secure communication is under investigation, and no theoretical analysis of impulsive spatiotemporal synchronization found to determine conditions of impulses to achieve desired property of synchronization, and the analysis of Lyapunov exponents of these models has not been explored.

The equi-attractivity property [50, 51], was used to investigate applying impulsive spatiotemporal synchronization between two continuous-time extended systems of PDE′s, and set up conditions of systems parameters with impulse durations and magnitudes. This theoretical mathematical development explained how and why impulsive synchronization of spatiotemporal chaotic systems works, compared with known numerical results about synchronization [52, 53, 54]. These theoretical results were confirmed by analyzing Lyapunov exponents of dynamic errors generated from impulsive synchronization of spatiotemporal chaotic systems [55, 56]. Generalize this technique to PDE′s by incorporating numerical method of lines [57, 58], and generate a numerical results representing a sufficient condition for impulsive synchronization which are consistent with theoretically analysis obtained from the same systems.

The aim of this work is to generalize impulsive control of spatiotemporal chaos with sufficient conditions for: non-linear Kuramoto–Sivashinsky PDE′s; and two identical one-dimensional Grey–Scott models for a diffusion reaction system using Lyapunov exponents to achieve equi-attractivity, stability of large chaotic systems with maintaining the chao approaching zero. The theoretical development of the theory, and remarks are concluded.

2. Preliminaries

Consider the impulsive initial boundary value problem presented by one-spatial dimension) nth order partial derivative equations given by:

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots, \frac{\partial^n u}{\partial x^n}\right)\ t \neq t_k,$$

$$t \in R_+, k = 1, 2, 3\ \dots$$

$$\Delta u(t,x) = Q_k u(t,x), t = t_k, \tag{1}$$

$$t \in R_+, k = 1, 2, 3\ \dots$$

$$u\ (0^+, x) = u_o(x), x \in [0^+, L]$$

$$u(t,0) = u\ (t,L) = h_1(t), t \in R_+$$

$$\frac{\partial u}{\partial x}\ (t,0) = \frac{\partial u}{\partial x}(\ t,L) = h_2(t), t \in R_+$$

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$$\frac{\partial^{n_1} u}{\partial x^{n_1}}(t,0) = \frac{\partial^{n_1} u}{\partial x^{n_1}}(\ t,L) = h_{n_1}(t), t \in R_+$$

For some $n_l \geq 0$, where

$$\frac{\partial u}{\partial t} = (\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \dots, \frac{\partial u_m}{\partial t})^T$$

$$\frac{\partial u}{\partial x} = (\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \dots, \frac{\partial u_m}{\partial t})^T$$

$$\frac{\partial^2 u}{\partial x^2} = (\frac{\partial^2 u_1}{\partial x^2}, \frac{\partial^2 u_2}{\partial x^2}, \dots, \frac{\partial^2 u_m}{\partial x^2})^T$$

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$$\frac{\partial^n u}{\partial x^n} = (\frac{\partial^n u_1}{\partial x^n}, \frac{\partial^n u_2}{\partial x^n}, \dots, \frac{\partial^n u_m}{\partial x^n})^T,$$

$$\Delta u_{(t_k, x)} = u\ (t_k^+, x) - u(t_k^-, x),$$

for all

$$x \in [o, L], u\ (t_k^+, x)\ =$$

$\lim_{t\to t_k^+} u(t,x)$, for a fixed $x \in [0, L]$, and the moment of impulse satisfying: $0 = t_1 < t_2 < \dots\dots < t_k < \dots$ and $\lim_{k\to\infty} t_k = \infty$,

The matrices Q_k are m x m constant matrices satisfying:

$||Q_k|| = \sqrt{\lambda_{max}(Q_k^T Q_k)} < L_1$, For every k = 1, 2, … and some $L_1 > 0$

($\lambda_{max}\ (Q^T Q)$ is the largest eigenvalue of $Q^T Q$). Let

f: $R_+ \times [0, L] \times R^m \times \dots \to R^m$ be continuous on $(t_k, t_{k+1}] \times [0, L] \times R^m$

$\times \dots \to R^m$, and $f(t_k^+, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots, \frac{\partial^n u}{\partial x^n})$ exists for every k = 1, 2, 3,…,

Let n= 2 in the above model, and assume that f satisfies

Lipschitz conditions with respect to: u, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$. Furthermore assume the existence of the functions f_1 (t, u), and f_2 (t, u) such that

$$f_1(t, u) \le f(t, x, u, 0, 0) \le f_2(t, u)$$

for every (t, x, u) $x \in [0,T]\ x\ [0,L]\ x\ R^m$ where inequality holds compo-nentwise, and T is a positive number, and that there exist solutions:

$\gamma(t)$, and $\rho(t)$ to the following systems:

$$\dot{\gamma}(t) = f_1(t,\gamma), t \neq 0, t_k, T, 1 \le k \le m_1$$

$$\Delta\gamma_{(t_k)} = Q_k\gamma(t_k), 1 \le k \le m_1$$

$$\gamma(O^+) = \gamma_o$$

and

$$\dot{\rho}(t) = f_2(t,\rho), t \neq 0, t_k, T, 1 \le k \le m_1$$

$$\Delta\rho_{(t_k)} = Q_k\,\rho(t_k), 1 \le k \le m_1$$

$$\rho(O^+) = \rho_o$$

respectively, where $t_{m_1} \le T$. If $\rho_o \le u_o(x) \le \gamma_o$ on [0, L] and if there exists a function:

$P \in C\big((0,T)\ \mathrm{x}\ \{0,L\}, R_+\big)$ such that, for i = 1, …, n_1

$$P(t,x)\rho\ (t) \le h_i(t) \le P(t,x)\gamma(t),$$

$t \neq t_k, k = 1,2,3,\dots\dots, m_1$, then there exists a local solutions$u(t,x)$: for system (1) satisfying

$$\rho\ (t) \le u(t,x) \le \gamma(t)$$

provided that the original partial differential equation, in (1), without the impulses, has a solution [59]. For $x \in [o,L]$, let $u(t,x) = u(t,x,u_o(x))$ be any solution of (1) satisfying:

$u(O^+,x) = u_o(x)$, and u(t, x) be left continuous at each $t_k > 0$, k = 1,2,..., in its interval of existence i.e.

$u(t_k{}^-,x) = u(t_k,x)$ For every $x \in [0,L]$

Definition 1. Suppose that: $u(t,x)$: R_+ x $[0,L] \to R^m$ for some $m > 0$, where u is of class $\ell_2[0,L]$ with respect to x. Then $\|.\|_2$ $of\ u(t,x)$ is defined by $\|u(t,x)\|_2 = \left\{\int_0^L \|u(t,x)\|^2\, dx\right\}^{\frac{1}{2}}$

where$\|.\|$ is Euclidean norm. For studying the dynamics of a particular systems whose structures resemble system (1), [60]

The following classes of functions, and definitions were discussed:

Let:

$$S^C(M) = \{u \in R^m : \|u\|_2 \ge M\}$$

$$S^C(M)^0 = \{U \in R^m : \|u\|_2 > M\}$$

$$v_o(M) = \{V: R_+\ \mathrm{x}\ S^C(M) \to R_+ : V(t,u) \in C\left((t_k,t_{k+1}]\mathrm{x}\, S^C(M)\right),$$

locally Lipschitz in u, and V (t_k^+, u) exists for k = 1, 2, …3}, where

M ≥ 0.

Definition 2: Let M ≥ 0, V∈ $v_o(M)$, define the upper right derivative of V(t, u) with respect to the continuous portion of the system (1), for: (t, u) ∈ R_+ x $S^C(M)^o, and\ t \neq t_k, k = 1,2,3,\dots, by$

$$D_t^+V(t,u) = \lim_{\delta\to 0^+} Sup\frac{1}{\delta}\,[\,V(t+\delta,u+\delta f\left(t,x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\dots,\frac{\partial^n u}{\partial x^n}\right)) - V(t,u)].$$

Definition 3: Solutions of the impulsive system (1) are said to be

(S1) equi-attractive in the large if for each $\varepsilon > 0, \alpha > 0, t_o \in R_+$,
there exists a number T = T $(t_o,\varepsilon,\alpha) > 0$ such that $\|u(t_o,x)\|_2 < \alpha$
implies $\|u(t_o,x)\|_2 < \varepsilon$, for t ≥ t_o+ T;

(S2) uniformly equi-attractive in large if T in (S1) is independent of t_0.

From the definition of equi-attractivity in the large, it can been seen that the solutions of system (1) possessthis property will approach zero with respect to $\|.\|_2$, no matter how large $\|u(t_o,x)\|_2$ is.
i.e. $\lim_{t\to\infty}\|u(t,x)\|_2 = 0$. Moreover, the properties (S_1), and (S_2) in Definition 3 become identical for autonomous system [61], i.e.
when:

$$f\left(t,x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\dots,\frac{\partial^n u}{\partial x^n}\right) = f\left(x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\dots,\frac{\partial^n u}{\partial x^n}\right).$$

Therefore when dealing with the autonomous systems, the uniform terminology will be automatically removed. The above definitions will be used heavily in exploring the conditions under which the solutions generated by several impulsive PDE′s are equi-attractive in the large. The following sections represented the generalization of impulsive control of Kuramoto–Sivashinsky PDE′s equation and Grey–Scott models for a diffusion reaction system; using Lyapunov exponents to study chaotic large systems, and maintaining the chaos nearly zero.

3. Impulsive Control of Kuramoto-Sivashinsky PDE′s Equation

These equations are represented by the impulsive initial boundary conditions:

$$u_t + u^2 x + a\,(x)u_{xx} + u_{xxxx} = 0, t \neq t_k\ t \in R_+, k = 1,2,3\ \dots \tag{2}$$

$$a\,(x) \ge \delta > 0, 0 < x < L$$

$$\Delta u(t,x) = -q_k u(t,x)\ t = t_k\ t \in R_+, k = 1,2,3\ \dots$$

$$u(0,x) = u_0(x), x \in [0,L]$$

$$u(t,0) = u(t,L) = 0, t \in R_+$$

$$u_x(t,0) = u_x\,(t,L) = 0, t \in R_+$$

Where $q_k > 0, k = 1,2,3,\ldots$, and L is the only free parameter. Equation (2) with absence of the impulses exhibited extensive chaos, indicating that Lyapunov dimension of the attractor grows linearly with the system volume size (L) [62]. The following Lemma gives the upper bounds on: $\|u(t,x)\|_2$, and $\|u_x(t,x)\|_2$ in terms of $\|u_{xx}(t,x)\|_2$. This Lemma is the well-known Poincare inequality [6, 33] lemmas.

Lemma 1: Let J = [0, L] and $M \in C^2(J)$. If u(0) = u(L) = 0, then

$$\|u(x)\|_2 \le \frac{L}{\pi} \|u_x(x)\|_2 \tag{3}$$

The next theorem gives the required criteria for system (2) to be equi-attractive in large., and achieved controlling chaotic behavior of Kuramoto-Sivashinsky by forcing the solution to converge to zero.

Theorem 1: Let q = $\min_k q_k$, and $\Delta_{k+1} = t_{k+1} - t_k \le \Delta$, *for k = 1, 2, 3,..., and for* some $\Delta > 0$. Then the impulsive Kuramoto-Sivashinsky equation (2) is equi-attractive in large if:

$$(1-q)^2\, e^{\Delta\mu} < 1, \text{ where } \delta > \frac{\pi^2}{L^2}$$

This theorem will be proved by choosing an appropriate Lyapunov function $V(u(t,x))$.

Let: $V(u(t,x)) = \|u(t,x)\|_2^2 = \int_0^L u(t,x)^2\,dx$

Using system (2) with its boundary conditions, definition 2, and applying integration by parts gave: $t \in (t_k, t_{k+1}]$, k = 1, 2, 3, …,

$$D_t^+ V(u(t,x)) = \int_0^L 2u\,(t,x)\,u_t(t,x)dx$$

$$= \int_0^L (-\,4u(t,x)^2\; u_x(\,t,x) - 2u(t,x)a(x)u_{xx}(t,x)$$

$$-2u(t,x)u_{xxxx}(t,x))dx$$

$$= -\frac{4}{3}u(t,x)^3\,\Big]_0^L - 2\int_0^L u(t,x)a(x)u_{xx}(t,x)dx$$

$$-2\int_0^L u(t,x)u_{xxxx}(t,x)dx$$

$$= -\frac{4}{3}[u(t,x)^3]_0^L - 2([\,u(t,x)a(x)u(t,x)]_0^L$$

$$-\int_0^L (u(t,x)a_x(x) + u_x\,(t,x)a(x))\,u_x(t,x)dx)$$

$$-2[u(t,x)u_{xxx}(t,x)]_0^L + 2[u(t,x)u_{xx}(t,x)\,]_0^L$$

$$-\,2\int_0^L u_{xx}(t,x)^2\,dx$$

$$= 2\int_0^L (u(t,x)a_x(x)u(t,x)_x\,dx \tag{4}$$

$$+2\int_0^L u_x(t,x)^2\,a_x(x)dx - 2\int_0^L u_{xx}(t,x)^2\,dx$$

$$=2[\,a(x)u(t,x)\,u_x\,(t,x)\,]_0^L -$$

$$\int_0^L (u(t,x)u_{xx}(t,x) + u_x(t,x)^2)a_x(x)dx$$

$$+2\int_0^L u_x(t,x)^2\,a_x(x)dx - 2\int_0^L u_{xx}(t,x)^2\,dx$$

$$= -2\int_0^L (u(t,x)\,u_{xx}\,(t,x) + u_x(t,x)^2\,a_x(x))dx + 2\int_0^L u_x(t,x)^2\,a_x(x)dx - 2\int_0^L u_{xx}(t,x)^2\,dx$$

$$\le -2\delta\int_0^L (u(t,x)\,u_{xx}\,(t,x)dx + \int_0^L u_x(t,x)^2\,dx + 2\delta\int_0^L u_x(t,x)^2\,dx - 2\int_0^L u_{xx}(t,x)^2\,dx$$

$$\le -2\delta\,[\,u(t,x)\,u_x\,(t,x)\,]_0^L - \int_0^L u_x(t,x)^2\,dx$$

$$+\int_0^L u_x(t,x)^2 dx + 2\delta\int_0^L u_x(t,x)^2\,dx - 2\int_0^L u_{xx}(t,x)^2\,dx$$

$$\le -2\delta\|u(t,x)\|_2^2 - \|u_{xx}(t,x)\|_2^2$$

However Lemma 1 gave rise to:

$$\|u_{xx}(t,x)\|_2^2 \le \frac{\pi^2}{L^2}\|u_x(t,x)\|_2^2$$

Since $u_x(t,x)$ satisfies the conditions of Lemma 1, the following condition be obtained:

$$\|u_x(t,x)\|_2^2 \le \frac{L^2}{\pi^2}\|u_{xx}(t,x)\|_2^2$$

Thus:

$$D_t^+ V\big(u(t,x)\big) \le 2\delta\frac{\pi^2}{L^2}\|u(t,x)\|_2^2 - 2\frac{\pi^4}{L^4}\|u(t,x)\|_2^2$$

$$\le 2\delta\,\frac{\pi^2}{L^2} - 2\frac{\pi^4}{L^4}\|u(t,x)\|_2^2 \le \mu\|u(t,x)\|_2^2$$

Hence, for all $t \in (t_k, t_{k+1}]$, k = 1, 2, 3, …, then we have:

$$V\big(u(t,x\,)\big) \le e^{\mu(t-t_k)}V\big(u(\,t_k^+,x\,)\big), \tag{5}$$

And (5)

$$V\big(u(\,t_{k+1},x\,)\big) \le e^{\mu\Delta_{k+1}}V\big(u(\,t_k^+,x\,)\big) \tag{6}$$

Moreover according to the structure of the impulses in system (2)

$$for\ all\ x \in [0,L\,], k = 1,2,3,\ldots,$$

$$u(t_k^+,x\,) = u\,(t_k,x) - q_k u(\,t_k\,,x\,) = (1-q_k)u(\,t_k\,,x) \Rightarrow$$

$$\int_0^L u(t_k^+,x)^2\,dx = \int_0^L (1-q_k)^2\,u\,(t_k,x)^2\,dx$$

It follows that:

$$V(u(t_k^+,x)) = (1-q_k)^2\,V(u\,(t_k,x)) \qquad (7)$$

Hence, by using inequalities (6), (7) we have for every, k = 1, 2, 3,…,

$$V\big(u(t_{k+1},x)\big) \le (1-q_k)^2 e^{\mu\Delta_{k+1}} V\big(u(t_k,x)\big)$$

$$\le (1-q_k)^2 e^{\mu\Delta} V\big(u(t_k,x)\big)$$

By inequalities (4), and (8). It can be concluded that $\lim_{k\to\infty} \mathrm{V}\left(\mathrm{u}(t_k,x)\right) = 0$ Therefore by inequality (5), we have for all: $t \in [t_k, t_{k+1}]$, k = 1, 2, 3,…,

$$V\big(u(t,x)\big) \le e^{\mu\Delta_{k+1}} V\left(u(t_k^+,x)\right) \le (1-q_k)^2\,e^{\mu\Delta_{k+1}}$$

$V\big(u(t_k,x)\big) \le (1-q)^2 e^{\mu\Delta} V\big(u(t_k,x)\big) \to 0$ as k $\to \infty$ It follows that:

$$\lim_{t\to\infty} V\big(u(\mathrm{t},x)\big) = 0$$

i.e. the solutions to impulsive Kuramoto-Sivashinsky equation, defined by system (2) are equi-attractive in the large. *Remark 1.* Form theorem 1, chaotic behavior of Kuramoto-Sivashinsky equations described by PDE'S in (2) reach stability state, and equi-attractivity property was achieved by using partial derivatives of PDE′S, and Lyapunov functions.

Remark 2. it was concluded from theorem 1, that if ratio chosen to be more accurate than Kuramoto-Sivashinsky equation, solutions will continue to be equi-attractive in a large, even with lack of impulses. However, impulses are required for stabilization the system, provided that impulses meeting the requirement set described in theorem 1.

Remark 3. A sufficient condition in theorem 1. It is not necessary. In other words. PDE's impulsive, described by system (2) remain equi- attractive in large even the case of inequality (4) not satisfied.

4. Impulsive Synchronization of the Grey-Scott Model

The impulsive control methods have been successfully used for controlling chaotic behavior of Kuramoto-Sivashinsky equation by making its solutions equi-attractive in the large, although the original PDE exhibited spatiotemporal chaotic behavior. The authors extended this work and investigated the impulsive synchronization of two identical spatiotemporal chaotic systems using Grey-Scott model used as the spatiotemporal generator [63]. The novel generalized theory is definitely applicable to synchronization of two Kuramoto-Sivashinsky equations, and any other spatiotemporal chaotic system of the same structure. Grey-Scott cubic auto-catalysis model is a reaction diffusion system corresponding to two irreversible steps exhibited mixed mode spatiotemporal chaos, and is described by the equations:

$$\frac{\partial u_1}{\partial t} = -u_1\,u_2^2 + a(1-u_1) + d_1 a(x)\nabla^2 u \qquad (8)$$

$$\frac{\partial u_2}{\partial t} = u_1 u_2^2 - (a+b)u_2 + d_2\nabla^2 u_2 \qquad (9)$$

Where b is the dimensionless rate constant of the second reaction, a is the dimensionless feed rate, and d_1, d_2 are the diffusion coefficients. In the following section, the impulsive synchronization of the one-dimensional version of this system with another identical system starting from different initial conditions is discussed. i.e. synchronization of the chaotic signal $u(t,x) = (u_1(t,x), u_2(t,x))^T$, is given by transmitter:

$$\frac{\partial u_1}{\partial t} = -u_1 u_2^2 + a(1-u_1) + d_1 a(x)\frac{\partial^2 u_1}{\partial x^2}\; t \in R_+$$

$$\frac{\partial u_2}{\partial t} = u_1 u_2^2 - (a+b)u_2 + d_2\frac{\partial^2 u_2}{\partial x^2}\; t \in R_+ \qquad (10)$$

$$a\,(x) \ge \delta > 0, 0 < x < 1$$

$$u(0,x) = u_o(x), x \in [0,L]$$

$$u(t,0) = u(t,L) = h(t), t \in R_+$$

With the chaotic signal: $v(t,x) = (v_1(t,x), v_2(t,x)^T$ given by:

$$\frac{\partial v_1}{\partial t} = -v_1 v_2^2 + a(1-v_1) + d_1 a(x)\frac{\partial^2 v_1}{\partial x^2}$$

$$t \ne t_k,\ \mathrm{k} = 1, 2, \ldots.$$

$$\frac{\partial v_2}{\partial t} = v_1 v_2^2 - (a+b)v_2 + d_2\frac{\partial^2 v_2}{\partial x^2} \qquad (11)$$

$$a\,(x) \ge \delta > 0, 0 < x < 1$$

$$v(0,x) = v_o(x), x \in [0,L]$$

$$v(t,0) = v(t,L) = \tilde{h}(t), t \in R_+$$

$$\Delta v(t,x) = -Q_k e(t,x), t = t_k,$$

$$x \in [0,L], \mathrm{k} = 1, 2, \ldots.$$

Where a, b, d_1, and d_2, are defined previously: $u_o(x), v_o(x)$ are the initial conditions, $h(t)$ is the periodic boundary condition for the transmitter system, $e(t,x) = u(t,x) - v(t,x)$. Q_k are constant matrices satisfying $\| Q_k\| < L_1$, for every k = 1, 2, 3,…, and some $L_1 > 0$.

The boundary condition $\widetilde{h}(t)$ described at the receiver is defined by:

$$\tilde{h}(t) = h(t) - \mathrm{g}(\mathrm{t})\{1 + \sum_{k=1}^{\infty}[\left(\|I + Q_{k-1}\|^{2k} - \|I + Q_{k-1}\|^{2(k-1)}\right)$$

$$H\ (t-t_k\ )]\}\qquad(12)$$

$g(t)\ =\ (g_1(t),\ g_2(t))^T\ \epsilon\ C^1\ (R_+), ||Q_k\ ||\ \le N$, For some $N > 0$, and for all $t \in R_+$, I is identity matrix, Q_o is defined to be zero matrix (i.e. Q_o= 0), $H\ (t-t_k\ )$, k = 1, 2, 3, …, is the alternative heaviside step function defined by:

$$H\ (t-t_k\ )=\begin{cases}0\ if\ t\ \le\ t_k\\ 1\ if\ t\ >t_k\end{cases}$$

According to equations (10), (12). The error system $e(t,x)$ will be
given by:

$$\frac{\partial e_1}{\partial t}=-u_1u_2^2+v_1v_2^2-ae_1\ +d_1a(x)\frac{\partial^2 e_1}{\partial x^2}$$

$$t\in R_+, k=1,2,3,\dots$$

$$\Delta e\ (t,x)=Q_k\ e(\ t,x), t=t_k\ , x\in[0,L], k=1,2,3,\dots\ (13)$$

$$a(x)\ \ge\ \delta>0, 0<x\ <1$$

$$e\ (0,x)=\ e_0(x), x\in[0,L]$$

$$e\ (t,0)=\ e\ (t,L)=\widetilde{\boldsymbol{H}}(t), t\in R_+$$

Where $e_o(\ t)=u_o(\ t)-v_o(\ t), and$

$$\widetilde{\boldsymbol{H}}(t)=g(t)\{1+\sum_{k=1}^{\infty}[(||I+Q_{k-1}||^{2k}-||\ I+Q_{k-1}||^{2(k-1)})$$

$$H\ (t-t_k\ )]\}$$

$$Notice\ thet\ if||\ I+Q_k||\ \le\ L_2\ <1, \text{for every}$$

$$k=1,2,3,\dots, then$$

This is a very important property which will be used in upcoming theory. Furthermore because u and v functions are both generated by spatiotemporal chaotic systems, It can be concluded immediately that they are both equi-banded [64,65]. This will be also be a very useful property helped in the proof of the next theorem. Using the above description, to explore the idea of impulsively synchronizing the two systems u, and v reduces to proving the error system (13) is equi-attractive in the large or that: $\lim_{t\to\infty}||e(t,x)||_2=0$. It could be stated now two lemmas reported in ([66], theorem 3.1, p. 45, and Corollary 2.2 P. 33, respectively), Two other ones were proved in order to establish several results needed in obtaining certain criteria for system (13) to be equi-attractive in the large.

Lemma 2. Let $p(t)\neq 0$, and $r(t)$ be given functions for $t=l, l+1, l+2, \dots,$ for some $l\in R_+U\ \{0\}$. Then:

a) The solutions of the equation $w(t+1)=p(t)w(t$are given by:

$$w(t)=w(l)\prod_{S=l}^{t-1}p(s)$$

b) All solutions of the equation $z(t+1)=p(t)z(t)+r(t)$are given by: $z(t)=w(t)[\sum\frac{r(t)}{Ew(t)}+C]$

Where $\sum$ is the n definite sum, $E$is the shift operator $Ez(t)=z(t+1)$,

C is an arbitrary constant, w(t) is any non-zero solution from part (a)

Lemma 3. $If\ n\in Z_+U\ \{0\}, then$

$$\sum t^n=\frac{1}{1+n}\ B_{n+1}(t)+C,$$

Where B_n are Bernoulli polynomials, for all $n\in Z_+U\ \{0\}$, and C is an arbitrary constant.

Lemma 4. Put $p(t)=q$, and $r(t)=\ kt^nq^{t-1}$ in Lemma 2, for all $t\ =\ 1,2,3,\dots(l=1)$, where $0\ <q<1$, $n\in Z_+U\ \{0\}$, and $k\in R_+$. Then $\lim_{t\to\infty}Z(t)=0$

Proof. From Lemmas 2, and 3. It can be concluded that $w(t)=q^tw(1)$, and

$$z(t)=w(t)\left[\sum\frac{r(t)}{Ew(t)}+C\right]$$

$$=\ q^tw(1)\ \left[\sum\frac{kt^nq^{t-1}}{q^{t+1}\ w(1)}+C\right]\ =$$

$$Kq^{t-2}w(t)\ \left[\sum t^n+p\right]=Kq^{t-2}\left[\sum\frac{1}{n+1}\ B_{n+1}(t)+p\right]$$

$$=Kq^{t-2}\left[\sum\frac{1}{n+1}\ B_{n+1}(t)+p\right]$$

$$=\frac{K}{n+1}B_{n+1}(t)\ q^{t-2}+pKq^{t-2}$$

Where C is an arbitrary constant, and $p=qw(1)C/K$. Notice that

$$\lim_{t\to\infty}pKq^{t-2}=0,\ \lim_{t\to\infty}P\frac{K}{n+1}B_{n+1}(t)\ q^{t-2}=0$$

(Since $0<q<1$, B_n are Bernoulli polynomials for all $n\in Z_+U\ \{0\}$, and because of L'Hôpital's rule applied $n+1$ times for the second limit). It follows that

$$\lim_{t\to\infty}z(t)=0$$

We can also prove that the type of the function chosen for $r(t)$ in lemma (4) satisfies:

$$\lim_{t\to\infty}r(t)\ =\ \lim_{t\to\infty}Kt^nq^{t-1}=0$$

For all $\in Z_+U\{0\}$, and applying L'Hôpital's rule n times.

Lemma 5. Let, $f(u_1,u_2\ )=\ u_1\,u_2^2$ be defined over the set:

$$S=\ \{(u_1,u_2\ )^T\in R^2\text{:}\ 0\ \le\ |u_1|\le\beta_1, \text{, and}\ 0\ \le\ |u_2|\le\beta_2\}.$$

Then the function f satisfies Lipschitz condition on S with Lipschitz constant given by $L_o=\beta_2\sqrt{\beta_1^2+4\,\beta_1}$. I*n other words,* for every $(u_1\,,u_2\,)^T, (v_1\,,v_2\,)^T\ \in S$. There is

$$|f(u_1,u_2\ )-\ f(v_1,v_2\ )\ |\le L_o||(u_1-v_1,u-v_2)||$$

Proof: Since S is compact, and convex subset of, R^2, and f has continuous partial derivatives on S, so by the Mean value theorem [33], for some $c\in\ (c_1,c_2\ )^T$ in the line segment joining $(u_1,u_2\ )^T, (v_1,v_2\ )^T$ which lies entirely in S.

$$|f(u_1,u_2)-f(v_1,v_2)| = \|\nabla f\big((c).\ (u_1-v_1, u_2-v_2\big)\|$$

$$\leq \|\nabla f(c)\|\|(u_1-v, u_2-v_2)\|$$

$$= \|(c_2^2, 2c_1c_2)\|\|(u_1-v_1, u_2-v_2)\| =$$

$$|c_2|\sqrt{c_2^2+4c_1^2}\,\|(u_1-v_1, u_2-v_2)\|$$

$$\leq \beta_2\sqrt{\beta_2^2+4\beta_1}\,\|(u_1-v_1,\ u_2-v_2)\|$$

as required.

The following theorem established specified the conditions required to guarantee the convergence of solution of system (13) to zero as $t\to\infty$

Theorem 2: Let q_k be the largest eigenvalue of $(I+Q_k)^T(I+Q_k)$, and $\Delta_{k+1} = t_{k+1}-t_k \leq \Delta$, for all k = 1, 2, 3,…, and for some $\Delta > 0$. In addition let: $q = Sup_k\, q_k$, d = max $(\delta d_1, d_2)$,

$$\beta_i = \max(Sup_{t\in R_+} u_i(t),\ Sup_{t\in R_+} v_i(t))$$

$$\zeta_i = Sup_{t\in R_+}\left|\frac{\partial e_i}{\partial x}(t,L)-\frac{\partial e_i}{\partial x}(t,0)\right|\ for\ i=1,2$$

$$\beta = 4\beta_2\sqrt{\beta_2^2+4{\beta_1}^2}-2a, F = 2d\zeta/\beta.\ \text{If}$$

$\widetilde{H}(t) = \big(\widetilde{H}_1(t), \widetilde{H}_2(t)\big)^T, \widetilde{H}(t) = \widetilde{H}_1(t)+\widetilde{H}_2(t)$ for all $t\in R_+$, and

$$qe^{\Delta\beta} < 1 \tag{14}$$

Then system (13) is equi-attractive in the large.

Proof: The proof of this theorem is similar to that of theorem 1. Choosing the Lyapunov energy function to be:

$$V(e(t,x)) = \int_0^L e^T(t,x)\,e(t,x)\ dx$$

$$= \int_0^L (e_1^2(t,x)+e_2^2(t,x))\,dx$$

In this case, by equation (13), and Lemma 5:

$$D_t^+V(e) = \int_0^L\left(e_1\frac{\partial e_1}{\partial x}+e_2\frac{\partial e_2}{\partial x}\right)dx$$

$$= 2\int_0^L[(-u_1u_2^2-v_1v_2^2)e_1-a{e_1}^2+d_1a(x)\,e_1\frac{\partial^2 e_1}{\partial x^2}$$

$$+(u_1u_2^2-v_1v_2^2)\,e_2-(a+b)e_2^2+d_2e_2\frac{\partial^2 e_2}{\partial x^2}]\,dx$$

$$\leq 2\int_0^L[\,|u_1u_2^2-v_1v_2^2||e_1|+|u_1u_2^2-v_1v_2^2||e_2|\,]dx$$

$$+2\int_0^L[-a\,e_1^2-(a+b)e_2^2\,]dx$$

$$+2\int_0^L\left[d_1a(x)\,e_1\frac{\partial^2 e_1}{\partial x^2}+d_2e_2\frac{\partial^2 e_2}{\partial x^2}\right]dx$$

$$\leq (4\beta_2\sqrt{\beta_2^2+4\beta_1^2}\int_0^L\|e\|^2-2\int_0^L[ae_1^2+(a+b)e_2^2]\,dx$$

$$+2\int_0^L\left[d_1a(x)\,e_1\frac{\partial^2 e_1}{\partial x^2}+d_2e_2\frac{\partial^2 e_2}{\partial x^2}\right]dx$$

$$\leq (4\beta_2\sqrt{\beta_2^2+4\beta_1^2}-2a)\|e\|_2^2$$

$$+2\int_0^L\left[d_1a(x)\,e_1\frac{\partial^2 e_1}{\partial x^2}+d_2e_2\frac{\partial^2 e_2}{\partial x^2}\right]dx$$

Applying integrations by parts:

First term:

$$\int_0^L\left[e_1\ a(x)\frac{\partial^2 e_1}{\partial x^2}\right]dx$$

$$= [\,e_1\ a(x)\frac{\partial e_1}{\partial x}]_0^L$$

$$-\int_0^L(\frac{\partial e_1}{\partial x}\ a(x)+\frac{\partial a}{\partial x}e_1)\frac{\partial e_1}{\partial x}dx$$

$$= [e_1a(x)\frac{\partial e_1}{\partial x}]_0^L-\int_0^L(\frac{\partial e_1}{\partial x})^2\,a(x)dx-\int_0^L e_1\frac{\partial a}{\partial x}\frac{\partial e_1}{\partial x}dx$$

$$= [e_1a(x)\frac{\partial e_1}{\partial x}]_0^L - \int_0^L(\frac{\partial e_1}{\partial x})^2\,a(x)dx-[e_1\frac{\partial e_1}{\partial x}\,a(x)]_0^L - \int_0^L(e_1\frac{\partial^2 e_1}{\partial x^2}+\left(\frac{\partial e_2}{\partial x}\right)^2)a(x)dx$$

$$= \int_0^L e_1\frac{\partial^2 e_1}{\partial x^2}\,a(x)dx \leq \delta\int_0^L e_1\frac{\partial^2 e_1}{\partial x^2}\,dx = \delta(\left[e\frac{\partial e_1}{\partial x}\right]_0^L - \int_0^L\left[(\frac{\partial e_1}{\partial x})^2\,dx)\ \leq 2\delta\,d_1\zeta_1\widetilde{H_1}(t)-2d_1\delta\|e_x\|_2^2\right]$$

Second term:

$$\int_0^L e_2\frac{\partial^2 e_2}{\partial x^2}\,dx = \left[e_2\frac{\partial e_2}{\partial x}\right]_0^L$$

$$-\int_0^L(\frac{\partial e_2}{\partial x})^2dx$$

$$\leq 2\,d_2\zeta_2\widetilde{H_2}(t)-2d_2\|e_x\|_2^2$$

Thus:

$$D_t^+V(e) \leq (4\beta_2\sqrt{\beta_2^2+4\beta_1^2}-2a)\|e\|_2^2 +$$

$$2\,d_1\delta\zeta_1\widetilde{H_1}(t)-2d_1\delta\|e_x\|_2^2+2\,d_2\zeta_2\widetilde{H_2}(t)-2d_2\|e_x\|_2^2$$

Where $\delta d_1 + d_2 = d$

$$D_t^+V(e) \leq (4\beta_2\sqrt{\beta_2^2+4\beta_1^2}-2a)\|e\|_2^2+2\,d\,\zeta\widetilde{H}(t)$$

$$-2d\|e_x\|_2^2$$

However, $2d\|e_x\|_2^2 \le 0$. Therefore, it can be concluded that:

$$D_t^+V(e) \le (4\beta_2\sqrt{\beta_2^2 + 4\,\beta_1^2}\; - 2a)\,\|e\|_2^2 + \;2\,d\,\zeta\widetilde{\mathrm{H}}(t)$$

$$\Leftrightarrow \;\; D_t^+V(e) \le \beta\|e\|_2^2 + \;2\,d\,\zeta\widetilde{\mathrm{H}}(t)$$

$$= \beta V(e) + 2\,d\,\zeta\widetilde{\mathrm{H}}(t) \Leftrightarrow D_t^+V(e) - \beta V(e) \;\le 2\,d\,\zeta\widetilde{\mathrm{H}}(t)$$

By multiplying both sides of the later inequality by $e^{-\beta t}$ gives rise:

$$e^{-\beta t}\;\; D_t^+V(e) - \beta\; e^{-\beta t}\; V(e) \le 2\,d\,\zeta\widetilde{\mathrm{H}}(t)\; e^{-\beta t}$$

$$\Leftrightarrow \; D_t^+[\;e^{-\beta t}\;V(e)] \le 2\,d\,\zeta\widetilde{\mathrm{H}}(t)\;e^{-\beta t}$$

It implies by the definition of $\widetilde{\mathrm{H}}(t)$, and for every $t \in (t_k, t_{k+1}]$, that

$$\int_{t_k}^{t} D_s^+\big[\;e^{-\beta s}\;V(e)\big]\;ds \;\le\; -Fe^{-\beta t}\;\widetilde{\mathrm{H}}(t) + \;Fe^{-\beta t_k}\;\widetilde{\mathrm{H}}(t)$$

Hence, for $t \in (t_k, t_{k+1}]$, we have

$$V(e(t,x)) \le e^{\beta\Delta_{k+1}}\;V(e(\,t_k^+,x)) + F\big(e^{\beta\Delta_{k+1}} - 1\big)\widetilde{\mathrm{H}}(t) \quad (15)$$

and

$$V(e(t_{k+1},x)) \le e^{\beta\Delta_{k+1}}\;V(e(\,t_k^+,x)) + F(e^{\beta\Delta_{k+1}} - 1)\widetilde{\mathrm{H}}(t) \quad (16)$$

On the other hand, for every $x \in [0, L], and\ every\ k =$ 1,2, ..., The structure of the impulses given in system (13),

$$e(\,t_k^+,x) = (I +\;Q_k)e\;(t_k\,,x) \Leftrightarrow V(e(\,t_k^+,x))$$

$$= \int_0^L e^T(\,t_k\,,x)(I +\;Q_k)^T\;(I +$$

$$Q_k)\;e\;(t_k\,,x)dx \;\Leftrightarrow V(e(\,t_k^+,x) \;\le$$

$$q_k\;\int_0^L e^T(\,t_k\,,x)\;e(t_k\,,x)dx$$

i.e.,

$$V(e(\,t_k^+,x) \le\; q_k V(e(\,t_k\,,x)) \quad (17)$$

Substituting the inequality (17) into the inequalities (15), and (16), gave:

$$V(e(t,x)) \le\; q_k e^{\beta\Delta_{k+1}}\;V(e(t_k,x)) + F(e^{\beta\Delta_{k+1}} - 1)\widetilde{\mathrm{H}}(t) \quad (18)$$

and

$$V(e(t_{k+1},x)) \le q_k e^{\beta\Delta_{k+1}}\;V(e(t_k,x)) + F\big(e^{\beta\Delta_{k+1}} - 1\big)\;\widetilde{\mathrm{H}}(t) \quad (19)$$

Let: $V_k = V(e(t_k,x))$, for every k = 1, 2, 3, … In this case by inequality (19), and for every k = 1, 2, 3, …

$$V_{k+1} \;\le q_k e^{\beta\Delta_{k+1}}\;V_k +\;F(e^{\beta\Delta_{k+1}} - 1)\widetilde{\mathrm{H}}(t)$$

$$\le qe^{\beta\Delta}\;V_k +\;F(e^{\beta\Delta} - 1)\;\widetilde{\mathrm{H}}(t)$$

Since: q_k < q < 1, for every k = 1, 2, 3, …,
it can be concluded that for every: $t \in (t_k, t_{k+1}]$

$$F(e^{\beta\Delta} - 1)\widetilde{\mathrm{H}}(t) \;\le\; Fqe^{k-1}$$

Hence:

$$V_{k+1} \;\le\; qe^{\beta\Delta}\,V_k + Fq^{k-1} \quad (20)$$

Define: $\aleph_1 = V_1\,, \aleph_{k+1} = q_k e^{\beta}\;\aleph_k + Fq^{k-1}, for\ k =$ 1,2,3, ...,

This implies, by inequality (20) and induction, that $V_k \le \aleph_k$ for all $k = 1,2,3,...$, However, by Lemma 4, and inequaility (14), we have: $\lim_{k\to\infty}(\aleph_k) = 0, i.e. \lim_{k\to\infty}(V_k) = 0$. Therefore

$$\lim_{k\to\infty} V(e(\,t_k,x)) = 0$$

Which, in turn, implies that, by inequality (18):

$$\lim_{t\to\infty} V(e(\,t,x)) = 0$$

In other words. Solutions to system (13), and. In addition, to the three remarks described previously, the following remarks must be added:

Remark 4. In theorem 2. Two-dimensional model Gray-Scott is also in system (9), although the more complicated than that because of use of theorem needed to assess the bilateral integration of Lyapunov energy functions.

Remark 5. Theorem 2, confirmed existence of matrices enhanced elimination the error system (13) to solve, and achieving stability, and estimating numerically ratio based on knowledge of other system parameters.

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References

[1] Liu XZ, Willms AR. Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft. MPE 1996; 2: 277–99.

[2] Mahmoud M. El- Borai, Wagdy G. Elsayed and Ragab M. Al-Masroub, Exact solutions of some nonlinear complex fractional partial differential equations, international Journal of Mathematics Trends and Technology. (IJMTT), April 2016, vol. 32, No. 1, 4-9.

[3] Mahmoud M. El- Borai, Wagdy G. Elsayed and Ragab M. Al-Masroub, Exact solutions for time fractional coupled Witham-Boer - Kaup equations via exp- function method, international research Journal of Engineering and Technology (IRJET), September 2015, vol. 2, Issue 6, 307-315.

[4] LeVeque RJ. Finite volume methods for hyperbolic problems. Cambridge University Press; 2002.

[5] Yang T, Yang CM, Yang LB. Control of Ro¨ ssler system to periodic motions using impulsive control methods. PhysLett A 1997; 232: 356–61.

[6] Yang T, Yang LB, Yang CM. Impulsive control of Lorenz system. Phys D 1997;110:18–24.

[7] Mahmoud M. El- Borai, Partial differential equations in an unbounded medium, Academia Nazionalle Dei Lincei Series VIII vol. IV December 6, 1973, 657-622.

[8] Erbe LH, Freedman HI, Liu XZ, Wu JH. Comparison principles for impulsive parabolic equations with applications to models of single species growth. J Austral Math SocSer B 1991; 32: 382–400.

[9] Stojanovski T, Kocarev L, Parlitz U. Driving and synchronizing by chaotic impulses. Phys Rev E 1996; 43 (9): 782–5.

[10] Yang T, Chua LO. Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication. IEEE Trans Circuits Syst I 1997; 44 (10): 976–88.

[11] Yang T, Chua LO. Impulsive control and synchronization of non-linear dynamical systems and application to secure communication. Int J Bifurcat Chaos 1997; 7 (3): 645–64.

[12] Yang T, Yang LB, Yang CM. Impulsive synchronization of Lorenz systems. PhysLett A 1997; 226 (6): 349–54

[13] M. M. El-Borai, W. G. El-Sayed and N. N. Khalefa, Solvability of some Nonliner Integral Functional Equations, Americ. J. Theor.& Appl. statis,2017; 6 (5-1): 8-12..

[14] Mahmoud M. El-Borai, Wagdy G. Elsayed. and Ragab M. Al-Masroub, Exact Solutions for Some Nonlinear Fractional Parabolic Equations, Inter. J. Adv. Eng. Res. (IJAER), Vlo. 10, No. III, Sep. 2015, 106–122.

[15] Li ZG, Wen CY, Soh YC, Xie WX. The stabilization and synchronization of Chua_s oscillators via impulsive control. IEEE Trans Circuits Syst I 2001; 48(11): 1351–5.

[16] Mahmoud M. EL-Borai, Wagdy G. El-Sayed, Eman Hamd Allah, Alaa A. El- Shorbagy, On some partial differential equations with operator coefficients and non-local conditions, life Science J. 2013; 10 (4), (3333-3336).

[17] Mahmoud M. El- Borai, Partial differential equations in an unbounded medium, Academia Nazionalle Dei Lincei Series VIII vol. IV December 6, 1973, 657-622.

[18] Lakshmikantham V, Bainov DD, Simeonov PS. Theory of impulsive differential equations. Singapore: World Scientific; 1989.

[19] Lakshmikantham V, Liu XZ. Stability analysis in terms of two measures. Singapore: World Scientific; 1993.

[20] Yang T. In: Impulsive control theory. Lecture notes in control and information sciences, vol. 272. Berlin: Springer-Verlag; 2001.

[21] Yang T. Impulsive systems and control: theory and applications. Huntington, New York: Nova Science Publishers, Inc.; 2001.

[22] Franceschini G, Bose S, Scho¨ ll E. Control of chaotic spatiotemporal spiking by time-delay auto-synchronization. Phys Rev E 1999; 60 (5): 5426–34.

[23] Bainov D, Kamont Z, Minchev E. On the stability of solutions of impulsive partial differential equations of first order. Adv Math Sci Appl 1996; 6 (2): 589–98.

[24] Mahmoud M. El- Borai, On the intial value problem for partial differential equations with operator coefficients, Int. J. of Math. And Mathematical Sciences, 3, 1980, pp. 103-111.

[25] LeVeque RJ. Finite volume methods for hyperbolic problems. Cambridge University Press; 2002.

[26] Liu XZ. Stability results for impulsive differential systems with applications to population growth models. DynamStabilSyst1994; 9 (2): 163–74.

[27] Mahmoud M. El- Borai, and Mohammed I. Abbas, On some integro-differential equations of fractional arders involving caratheodory nonlinearities, Internationa J. of Modern Mathematics, 2 (1) (2007), 41-52.

[28] Kelley WG, Peterson AC. Difference equations: an introduction with applications. 2nd ed. Academic Press; 2001.

[29] Mahmoud M. El- Borai, and Khairia El- Said El- Nadi, A parabolic transform and some stochastic ill- posed problems, British Journal of Mathematics and computer Science, 9 (5), 418-426, 2015.

[30] Mahmoud M. El- Borai, and Khairia El- Said El- Nadi, Fractional integrated semi groups and nonlocal fractional differential equations, Journal of Progressive Research in Mathematics (JPRM), vol. 5, Issue 3, September 22, 2015, 553-561.

[31] Boccaletti S, Bragard J, Arecchi FT. Controlling and synchronizing space time chaos. Phys Rev E 1999; 59 (6): 6574–8.

[32] Boccaletti B, Bragard J, Arecchi FT, Mancini H. Synchronization in non-identical extended systems. Phys Rev Lett 1999;83(3):536–9.

[33] Mahmoud M. El- Borai, Wagdy G. Elsayed and Ragab M. Al-Masroub, Exact solutions for some nonlinear fractional parabolic equations, International Journal of Advances in Engineering Research, vol. 10, Issue III, September 2015, 106-122.

[34] K. E. El-Nadi, W. G. El-Sayed & A. K. Qassem, On Some Dynamical Systems of Controlling Tumor Growth, Inter. J. App. Sci. & Math.(IJASM), Vol. 2 (5), (2015), 146-151.

[35] Mahmoud M. El – Boria, Generalized random processes and Cauchy's problem for some partial differential equations, Int. J of Math. And mathematical Sc. No. 3, 1980, pp. 549-558.

[36] Junge L, Parlitz U. Synchronization and control of coupled Ginzburg–Landau equations using local coupling. Phys Rev E 2000; 61 (4): 3736–42.

[37] Junge L, Parlitz U, Tasev Z, Kocarev L. Synchronization and control of spatially extended systems using sensor coupling. Int J Bifurcat Chaos 1999; 9 (12): 2265–70.

[38] Tasev Z, Kocarev L, Junge L, Parlitz U. Synchronization of Kuramoto–Sivashinsky equations using spatially local coupling. IntJ Bifurcat Chaos 2000; 10 (4): 869–73.

[39] M. M. El-Boraie, W. G. El-Sayed & R. M. Al-Masroubn Exact solutions of some non linear complex fractional partial differential equations, Inter. J. Math. Tre. & Tech. (IJMTT) – Vol. 32, No. 1, April (2016), 4-9.

[40] Mahmoud M. El- Borai, Wagdy G. Elsayed and Faez N. Ghaffoori, On the Cauchy problem for some parabolic fractional differential equations with time delays, Journal of Mathematics and system Science, 6, 2016, 194-199.

[41] M. M. El-Boraie, W. G. El-Sayed & R. M. Al-Masroubn, Exact Solutions of Some Nonline Partial Differential Equations via Extended (G'/G)-Expansion Method, Inter. J. Math. Trends and Tech. (IJMTT)– Vol. 36, No.1 Aug.2016, 60-71.

[42] Mahmoud M. El- Borai, The fundamental solutions for fractional evolution equations of parabolic type, J. of Appl. Math. Stochastic Analysis (JAMSA) 2004, 199-211.

[43] Mahmoud M. El- Borai, and Khairia El- Said El- Nadi, Osama Labib, Hamdy M., Volterra equations with fractional stochastic integrals, Mathematical problems in Engineering, 5, (2004), 453-468.

[44] Mahmoud M. El- Borai, Wagdy G. Elsayed and Ragab M. Al-Masroub, Exact solutions of some nonlinear partial differential equations via Extended – expansion method, international Journal of Mathematics Trends and Technology, (IJMTT), August 2016, vol 36, No. 1, 60-71.

[45] Fraser B, Yu P, Lookman T. Secure communications using chaos synchronization. Phys Canada, Special Issue on Non-Linear Dynam 2001; 57 (2): 155–61.

[46] Garcı´a-Ojalvo J, Roy R. Spatiotemporal communication with synchronized optical chaos. Phys Rev lett 2001; 86: 5204–7.

[47] Mahmoud M. El- Borai, Wagdy G. Elsayed and Faez N. Ghaffoori, On the solvability of nonlinear integral functional equation, international Journal of Mathematics Trends and Technology, (IJMTT), June 2016, vol. 34, No., 1, 39-44.

[48] Mahmoud M. El- Borai, Wagdy G. Elsayed and Faez N. Ghaffoori, Existence Solution for a fractional nonlinear integral equation of Volterra type, Aryabhatta Journal of Mathematics and informatics, July 2016, vol. 8, Issue 2, 1-15.

[49] Kocarev L, Tasev Z, Parlitz U. Synchronizing spatiotemporal chaos of partial differential equations. Phys Rev Lett 1997; 79 (1): 51–4.

[50] Kocarev L, Tasev Z, Stojanovski T, Parlitz U. Synchronizing spatiotemporal chaos. Chaos 1997; 7 (4): 635–43.

[51] M. M. El-Boraie, W. G. El-Sayed & F. N. Ghaffoori, On the Cauchy problem for some parabolic fractional partial differential equationswith time delays, J. Math. & System Scie, 6 (2016) 194-199.

[52] Itoh M, Yang T, Chua LO. Conditions for impulsive synchronization of chaotic and hyperchaotic systems. Int J Bifurcat Chaos 2001; 11 (2): 551–60.

[53] Mahmoud M. El- Borai, Wagdy G. Elsayed Adham M. Jawad, Adomian decomposition method for solving fractional differential equations, international Research Journal of Engineering and Technology (IRJET) vol. 2, Issue: 6 September 2015, 296-306.

[54] Yang T. In: Impulsive control theory. Lecture notes in control and information sciences, vol. 272. Berlin: Springer-Verlag; 2001.

[55] Khairia El-Said El-Nadi, L. M. Fatehy, Norhan Hamdy Ahmed, Marshall-okin Exponential Pareto Distribution with Application on Cancer Stem Cells, Americ. J. Theor.& Appl. statis, 2017; 6 (5-1): 1-7.

[56] M. M. El- Borai, H. M. El- Owaidy, Hamdy M. Ahmed and A. H. Arnous, Seithuti Moshokao, Anjan Biswas, Milivoj Belic, Dark and singular optical solitons solutions with saptio-temporal dispersion, Accepted, Internationa Journal for Light and Electron Optics, November 2016.

[57] M. M. El- Borai, H. M. El- Owaidy, Hamdy M. Ahmed and A. H. Arnous, Exact and Soliton Solutions to nonlinear transmission line model, Nonlinear Dyn. Published online: 16 September 2016.

[58] Cross MC, Hohenberg PC. Pattern formation outside of equilibrium. Rev Mod Phy 1993; 65 (8): 851–1112.

[59] Liu XZ, Willms AR. Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft. MPE 1996; 2: 277–99.

[60] Garcı´a-Ojalvo J, Roy R. Parallel communication with optical spatiotemporal chaos. IEEE Trans Circuits Syst I 2001; 48 (12): 1491–7.

[61] Anmar Khadra, Xinzhi Liu, Xuemin Shen. impulsive control and synchronization of spatiotemporal chaos, Chaos, Solitons and Fractals; 26 (2005), 615-636.

[62] LeVeque RJ. Numerical methods for conservation laws. Basel: Birkha¨userVerlag; 1990.